ELECTRIC CIRCUIT THEORY
AND
THE OPERATIONAL CALCULUS

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OPERATIONAL CALCULUS

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PREFACE

The following pages embody, substantially as delivered, a course of fifteen lectures given during the Spring of 1925 at the Moore School of Electrical Engineering of the University of Pennsylvania.

After a brief introduction to the subject of electric circuit theory, the first five chapters are devoted to a systematic and fairly complete exposition and critique of the Heaviside Operational Calculus, a remarkably direct and powerful method for the solution of the differential equations of electric circuit theory.

The name of Oliver Heaviside is known to engineers the world over: his operational calculus, however, is known to, and employed by, only a relatively few specialists, and this notwithstanding its remarkable properties and wide applicability not only to electric circuit theory but also to the differential equations of mathematical physics. In the writer's opinion this neglect is due less to the intrinsic difficulties of the subject than to unfortunate obscurities in Heaviside's own exposition. In the present work the *operational calculus* is made to depend on an integral equation from which the Heaviside Rules and Formulas are simply but rigorously deducible. It is the hope of the writer that this mode of approach and exposition will be of service in securing a wider use of the operational calculus by engineers and physicists, and a fuller and more just appreciation of its unique advantages.

The second part of the present work deals with advanced problems of electric circuit theory, and in particular with the theory of the propagation of current and voltage in electrical transmission systems. It is hoped that this part will be of interest to electrical engineers generally because, while only a few of the results are original with the present work, most of the transmission theory dealt with is to be found only in scattered memoirs, and there accompanied by formidable mathematical difficulties.

While the method of solution employed in the second part is largely that of the operational calculus, I have not hesitated to employ developments and extensions not to be found in Heaviside. For example, the formulation of the problem as a Poisson integral equation is an original development which has proved quite useful in the actual numerical solution of complicated problems. The same may be said of the Chapter on Variable Electric Circuit Theory.

In view of its two-fold aspect this work may therefore be regarded either as an exposition and development of the operational calculus with applications to electric circuit theory, or as a contribution to advanced electric circuit theory, depending on whether the reader's viewpoint is that of the mathematician or the engineer.

In the text I have not made reference to the literature of the subject, now fairly extensive, except where the treatment in the text is inadequate and I wish to refer the reader to a fuller and more satisfactory discussion. In the bibliography, however, there is furnished a list of original papers and memoirs. While this bibliography includes all pertinent papers known to the author, no claim to completeness can be made and it is possible that important papers have been omitted. The author would be grateful to any reader calling his attention to such omissions.

It remains a pleasurable duty to acknowledge the assistance I have received in the preparation of this book. To Dr. Harold Pender, Dean of the Moore School of Electrical Engineering of the University of Pennsylvania I owe the invitation to deliver the lectures out of which this book grew, and stimulating interest and advice throughout their course. To my associate Mr. R. S. Hoyt I am indebted for a careful reading of the manuscript, the verification of most of the formulas, and valuable contributions and advice. For example, the extremely compact proof of formula (29) is a modification and improvement due to Mr. Hoyt of an earlier proof. To my assistant, Miss Dorothy Angell are due all numerical calculations and curves and the careful proofreading of the entire book.

Finally to the Officials of the American Telephone and Telegraph Company and in particular to Mr. O. B. Blackwell, Transmission Development Engineer, I would express appre-

ciation of their generous attitude toward the research underlying this book and my indebtedness for the opportunity and encouragement to put this work into shape for publication.

JOHN R. CARSON.

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ELECTRIC CIRCUIT THEORY AND THE OPERATIONAL CALCULUS

CHAPTER I

THE FUNDAMENTALS OF ELECTRIC CIRCUIT THEORY

While a knowledge, on the reader's part, of the elements of electric circuit theory will be assumed, it seems well to start with a brief review of the fundamental physical principles of circuit theory, the mode of formulating the equations, and some general theorems which will prove useful subsequently.

Electric Circuit Theory is that branch of general electromagnetic theory which deals with electrical oscillations in linear electrical networks. An *electrical network* is a connected set of circuits or meshes each of which can be regarded as made up of resistance, inductance, and capacitance elements. If the number of independent closed meshes or circuits is finite, the network is said to have a finite number of degrees of freedom equal to the number of meshes, and the problem is mathematically formulated by a finite number of simultaneous linear differential equations. If the number of closed circuits or meshes is made infinite and the size of the elements infinitesimal the problem is formulated by a partial differential equation, such as the *telegraph equation*.

There are two distinct problems involved in electric circuit theory. First, the derivation of the differential equations from the physical laws and the constants and connections of the system, that is, the mathematical formulation of the physical problem: and secondly the solution of the differential equations.

The equations of circuit theory may be established in a number of different ways. For example, they may be based on Maxwell's

dynamical theory. In accordance with this method, the network forms a dynamic system in which the currents play the rôle of velocities. If we therefore set up the expressions for the kinetic energy, potential energy and dissipation, the network equations are deducible from general dynamic equations.

The simplest, and for our purposes, a quite satisfactory basis for the equations of circuit theory are found in Kirchhoff's Laws. These laws state that

1. The total impressed force taken around any closed loop or circuit in the network is equal to the potential drop due to (a) resistance, (b) inductive reaction and (c) capacitive reactance.

2. The sum of the currents entering any branch point in the network is always zero.

Let us now apply these laws to an elementary circuit in order to deduce the physical significance of the circuit elements.

Consider an elementary circuit consisting of a resistance element R, an inductance element L and a capacity element C in series, and let an electromotive force E be applied to this circuit. If I denote the current in the circuit, the resistance drop is RI, the inductance drop is LdI/dt and the drop across the condenser is Q/C where Q is the charge on the condenser. It is evident that Q and I are related by the equation $I=dQ/dt$ or $Q=\int Idt$. Now apply Kirchhoff's law relating to the drop around the circuit: it gives the equation

$$RI+LdI/dt+Q/C=E.$$

Multiply both sides by I: we get

$$RI^2+\frac{d}{dt}\frac{1}{2}LI^2+\frac{d}{dt}\frac{Q^2}{2C}=EI.$$

The right hand side is clearly the rate at which the impressed force is delivering energy to the circuit, while the left hand side is the rate at which energy is being absorbed by the circuit. The first term RI^2 is the rate at which electrical energy is being converted into heat. Hence the resistance element may be defined as a device for converting electrical energy into heat. The second term $\frac{d}{dt}\frac{1}{2}LI^2$ is the rate of increase of the magnetic energy. Hence the inductance element is a device for storing energy in the magnetic field. The third term $\frac{d}{dt}Q^2/2C$ is the

rate of increase of the electric energy. Hence the condenser is a device for storing energy in the electric field.

In the foregoing we have isolated and idealized the circuit elements. Actually, of course, every circuit element dissipates some energy in the form of heat and stores some energy in the magnetic field and some in the electric field. The analysis of the actual circuit element, however, into three ideal components is quite convenient and useful, and should lead to no misconception if properly interpreted.

Now consider the general form of network possessing n independent meshes or circuits. Let us number these from 1 to n, and let the corresponding mesh currents be denoted by I_1, $I_2 \ldots I_n$. Let electromotive forces $E_1, E_2 \ldots E_n$ be applied to the n meshes or circuits respectively. Let L_{jj}, R_{jj}, C_{jj} denote the total inductance, resistance and capacity in series in mesh j and let L_{jk}, R_{jk}, C_{jk} denote the corresponding mutual elements between circuits j and k. Now write down Kirchhoff's equation for any circuit or mesh, say mesh 1; it is

$$\left(L_{11}\frac{d}{dt}+R_{11}+\frac{1}{C_{11}}\int dt\right)I_1+\left(L_{12}\frac{d}{dt}+R_{12}+\frac{1}{C_{12}}\int dt\right)I_2+ \\ \cdots +\left(L_{1n}\frac{d}{dt}+R_{1n}+\frac{1}{C_{1n}}\int dt\right)I_n=E_1$$

Corresponding equations hold for each and every one of the n meshes of the network. Writing them all down, we have the system of equations

$$\begin{aligned}
&\left(L_{11}\frac{d}{dt}+R_{11}+\frac{1}{C_{11}}\int dt\right)I_1+\cdots+\left(L_{1n}\frac{d}{dt}+R_{1n}+\frac{1}{C_{1n}}\int dt\right)I_n=E_1 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (1)\\
&\left(L_{n1}\frac{d}{dt}+R_{n1}+\frac{1}{C_{n1}}\int dt\right)I_1+\cdots+\left(L_{nn}\frac{d}{dt}+R_{nn}+\frac{1}{C_{nn}}\int dt\right)I_n=E_n
\end{aligned}$$

The system of simultaneous differential equations (1) constitutes the canonical equations of electric circuit theory. The interpretation and solution of these equations constitute the subject of Electric Circuit Theory, and it is in connection with their solution that we find the most direct and logical introduction to the Operational Calculus.

As an example of the appropriate mode of setting up the circuit equations, consider the two mesh network shown in Fig.

1. Writing down Kirchhoff's Law for meshes 1 and 2, respectively, we have

$$\left(L_1\frac{d}{dt}+R_1+\frac{1}{C_1}\int dt\right)I_1+M\frac{d}{dt}I_2=E_1$$
$$M\frac{d}{dt}I_1+\left(L_2\frac{d}{dt}+R_2+\frac{1}{C_2}\int dt\right)I_2=E_2.$$

In this case the self and mutual coefficients are given by

$$\begin{array}{lll} L_{11}=L_1 & L_{22}=L_2 & L_{12}=L_{21}=+M \\ C_{11}=C_1 & C_{22}=C_2 & C_{12}=C_{21}=0 \\ R_{11}=R_1 & R_{22}=R_2 & R_{12}=R_{21}=0. \end{array}$$

The conventions adopted for the positive directions of currents and voltages are indicated by the arrows. The sign of the mutual inductance M will depend on the relative mode of winding of the two coils.

Now write down Kirchhoff's Law, or the circuital equations for the network of Fig 2. They are

$$\left\{(L_1+L_3)\frac{d}{dt}+(R_1+R_3)+\left(\frac{1}{C_1}+\frac{1}{C_3}\right)\int dt\right\}I_1$$
$$-\left(L_3\frac{d}{dt}+R_3+\frac{1}{C_3}\int dt\right)I_2=E_1,$$
$$-\left(L_3\frac{d}{dt}+R_3+\frac{1}{C_3}\int dt\right)I_1$$
$$+\left\{(L_2+L_3)\frac{d}{dt}+(R_2+R_3)+\left(\frac{1}{C_2}+\frac{1}{C_3}\right)\int dt\right\}I_2=E_2.$$

Comparison with equations (1) shows that

$$\begin{array}{lll} L_{11}=L_1+L_3 & L_{22}=L_2+L_3 & L_{12}=L_{21}=-L_3 \\ R_{11}=R_1+R_3 & R_{22}=R_2+R_3 & R_{12}=R_{21}=-R_3 \\ \dfrac{1}{C_{11}}=\dfrac{1}{C_1}+\dfrac{1}{C_3} & \dfrac{1}{C_{22}}=\dfrac{1}{C_2}+\dfrac{1}{C_3} & \dfrac{1}{C_{12}}=\dfrac{1}{C_{21}}=-\dfrac{1}{C_3}. \end{array}$$

It should be observed that the signs of the mutual coefficients R_{12}, L_{12}, C_{12} are a matter of convention. For example if the conventional directions of I_2 and E_2 are reversed, the signs of the mutual coefficients are reversed.

The system of equations (1) possesses two important properties which are largely responsible for the relative simplicity of classical electric circuit theory. First, the equations are linear in both currents and applied electromotive forces. Secondly, the

coefficients L_{jk}, R_{jk}, C_{jk} are constants. Important electrotechnical problems exist, in which these properties no longer obtain. The solution, however, for the restricted system of linear equations with constant coefficients is fundamental and its solution can be extended to important problems involving non-linear relations and variable coefficients. These extensions will be taken up briefly in a later chapter.

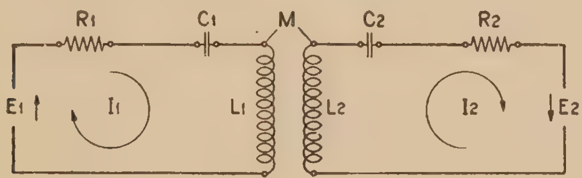

FIG. 1.

Another important property is the reciprocal relation among the coefficients; that is $L_{jk}=L_{kj}$, $R_{jk}=R_{kj}$, and $C_{jk}=C_{kj}$. It is easily shown that these reciprocal relations mean that there are no concealed sources or sinks of energy. Again important cases exist where the reciprocal relations do not hold. Such exceptions, however, while of physical interest do not affect the mathe-

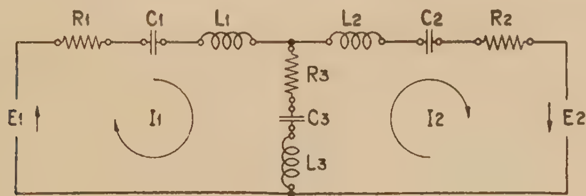

FIG. 2.

matical methods of solution, to which the reciprocal relation is not essential.

Returning to equation (1) we shall now derive the *equation of activity*. Multiply the first equation by I_1, the second by I_2, etc. and add: we get

$$\frac{d}{dt}\sum\sum\frac{1}{2}L_{jk}I_jI_k+\frac{d}{dt}\sum\sum\frac{1}{2}\frac{1}{C_{jk}}Q_jQ_k+\sum\sum R_{jk}I_jI_k=\sum E_jI_j. \qquad (2)$$

The right hand side is the rate at which the applied forces are supplying energy to the network. The first term on the left is the rate of increase of the magnetic energy

$$\frac{1}{2}\sum\sum L_{jk}I_jI_k,$$

while the second term is the rate of increase of the electric energy

$$\frac{1}{2}\sum\sum \frac{1}{C_{jk}}Q_jQ_k.$$

The last term, $\sum\sum R_{jk}I_jI_k$, is the rate at which electromagnetic energy is being converted into heat in the network. Consequently in the electrical network, the magnetic energy is a homogeneous quadratic function of the currents, the electric energy is a homogeneous quadratic function of the charges, and the rate of dissipation is a homogeneous quadratic function of the currents. In Maxwell's dynamical theory of electrical networks, these relations were written down at the start and the circuit equations then derived by an application of Lagrange's dynamic equations to the homogeneous quadratic functions.

Returning to equations (1), we observe that, due to the presence of the integral sign, they are integro-differential equations. They are, however, at once reducible to differential equations by the substitution $I = dQ/dt$, whence they become

$$\begin{aligned}
&\left(L_{11}\frac{d^2}{dt^2}+R_{11}\frac{d}{dt}+S_{11}\right)Q_1+\cdots+\left(L_{1n}\frac{d^2}{dt^2}+R_{1n}\frac{d}{dt}+S_{1n}\right)Q_n=E_1,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (3)\\
&\left(L_{n1}\frac{d^2}{dt^2}+R_{n1}\frac{d}{dt}+S_{n1}\right)Q_1+\cdots+\left(L_{nn}\frac{d^2}{dt^2}+R_{nn}\frac{d}{dt}+S_{nn}\right)Q_n=E_n.
\end{aligned}$$

Here, as a matter of convenience, we have written $1/C_{jk} = S_{jk}$. It is often more convenient, at least at the outset, to deal with equations (3) rather than (1).

The Exponential Solution.—In taking up the mathematical solution of equations (1), we shall start with the *exponential solution*. This is of fundamental importance, both theoretically and practically. It serves as the most direct introduction to the Heaviside operational calculus, and in addition furnishes the basis of the *steady state* solution, or the theory of alternating currents.

To derive this solution we set $E_1 = F_1e^{\lambda t}$ and put all the other forces $E_2, \ldots E_n$ equal to zero. This latter restriction is a

mere matter of convenience, and, in virtue of the linear character of the equations, involves no loss of generality.

Now, corresponding to $E_1 = F_1 e^{\lambda t}$, let us assume a solution of the form

$$I_j = J_j e^{\lambda t} \qquad (j = 1, 2 \ldots n)$$

where J_j is a constant. So far this is a pure assumption, and its correctness must be verified by substitution in the differential equations.

Now if $I_j = J_j e^{\lambda t}$, it follows at once that

$$\frac{d}{dt} I_j = \lambda I_j = \lambda J_j e^{\lambda t}$$

and

$$\int I_j \, dt = \frac{1}{\lambda} I_j = \frac{1}{\lambda} J_j e^{\lambda t}.$$

Now substitute these relations in equations (1) and cancel the common factor $e^{\lambda t}$. We then get the system of simultaneous equations:

$$\begin{aligned} (\lambda L_{11} + R_{11} + 1/\lambda C_{11}) J_1 + \cdots + (\lambda L_{1n} + R_{1n} + 1/\lambda C_{1n}) J_n &= F_1, \\ (\lambda L_{21} + R_{21} + 1/\lambda C_{21}) J_1 + \cdots + (\lambda L_{2n} + R_{2n} + 1/\lambda C_{2n}) J_n &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (\lambda L_{n1} + R_{n1} + 1/\lambda C_{n1}) J_1 + \cdots + (\lambda L_{nn} + R_{nn} + 1/\lambda C_{nn}) J_n &= 0. \end{aligned} \tag{4}$$

We note that this is a system of simultaneous *algebraic* equations from which the time factor has disappeared. It is this that makes the exponential solution so simple, since we can immediately pass from differential equations to algebraic equations. In these algebraic equations, n in number, there are n unknown quantities $J_1 \ldots J_n$. These can therefore all be uniquely determined. We thus see that the assumed form of solution is possible.

The notation of equations (4) may be profitably simplified as follows: write

$$\lambda L_{jk} + R_{jk} + 1/\lambda C_{jk} = z_{jk}(\lambda) = z_{jk}$$

and we have

$$\begin{aligned} z_{11} J_1 + z_{12} J_2 + \cdots + z_{1n} J_n &= F_1, \\ z_{21} J_1 + z_{22} J_2 + \cdots + z_{2n} J_n &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ z_{n1} J_1 + z_{n2} J_2 + \cdots + z_{nn} J_n &= 0. \end{aligned} \tag{5}$$

The solution of this system of equations is

$$J_j = \frac{M_{j1}(\lambda)}{D(\lambda)} F_1 = \frac{M_{j1}}{D} F_1$$

and

$$I_j = \frac{M_{j1}}{D} F_1 e^{\lambda t} = \frac{F_1}{Z_{j1}} e^{\lambda t} \tag{6}$$

where D is the determinant of the coefficients,

$$\begin{vmatrix} z_{11} & z_{12} & z_{13} \cdots\cdots & z_{1n} \\ z_{21} & z_{22} & z_{23} \cdots\cdots & z_{2n} \\ z_{31} & z_{32} & \cdots\cdots\cdots & z_{3n} \\ \cdots & \cdots & \cdots\cdots\cdots & \cdots \\ z_{n1} & z_{n2} & \cdots\cdots\cdots & z_{nn} \end{vmatrix} \tag{7}$$

and M_{j1} is the cofactor, or minor with proper sign, of the jth column and first row.

I shall not attempt to discuss the theory of determinants on which this solution is based.[1] We may note, however, one important property. Since $z_{jk} = z_{kj}$, $M_{jk} = M_{kj}$. From this the Reciprocal Theorem follows immediately. This may be stated as follows:

If a force $Fe^{\lambda t}$ is applied in the jth mesh, or branch, of the network, the current in the kth mesh, or branch, is by the foregoing

$$\frac{M_{kj}}{D} Fe^{\lambda t}.$$

Now apply the same force in the kth mesh, or branch, then the current in the jth mesh is

$$\frac{M_{jk}}{D} Fe^{\lambda t}.$$

Comparing these expressions and remembering that $M_{kj} = M_{jk}$, it follows that the current in the kth branch corresponding to an exponential impressed e.m.f. in the jth branch, is equal to the current in the jth branch corresponding to the same e.m.f. in the kth branch. This relation is of the greatest technical importance.

In many important technical problems we are interested only in two accessible branches, such as the sending and receiving. In such cases, where we are not concerned with the currents in the other meshes or branches, it is often convenient to eliminate

[1] For a remarkably concise and complete discussion of the exponential solution by aid of the theory of determinants, see Cisoidal Oscillations, Trans. A. I. E. E., 1911, by G. A. Campbell.

them from the equation. Thus, suppose that we have electromotive forces E_1 and E_2 in meshes 1 and 2 and are concerned only with the currents in these meshes. If we solve equations 3, 4, . . n, $n-2$ in number, for I_3 . . I_n in terms of I_1 and I_2 and then substitute in (1) and (2) we get

$$\begin{aligned} Z_{11}I_1+Z_{12}I_2&=E_1,\\ Z_{21}I_1+Z_{22}I_2&=E_2. \end{aligned} \tag{8}$$

The Steady State Solution.—The steady state solution, on which the whole theory of alternating currents depends, is immediately derivable from the exponential solution. Let us suppose that

$$E_2=E_3=\cdot\cdot\cdot=E_n=0 \text{ and that } E_1=F\cos(\omega t-\theta).$$

Now by virtue of the well known formula in the theory of the complex variable, $\cos x=\frac{1}{2}e^{ix}+\frac{1}{2}e^{-ix}$, we can write

$$\begin{aligned} E_1&=\tfrac{1}{2}Fe^{i(\omega t-\theta)}+\tfrac{1}{2}Fe^{-i(\omega t-\theta)},\\ &=\tfrac{1}{2}(\cos\theta-i\sin\theta)Fe^{i\omega t}+\tfrac{1}{2}(\cos\theta+i\sin\theta)Fe^{-i\omega t},\\ &=\tfrac{1}{2}F'e^{i\omega t}+\tfrac{1}{2}F''e^{-i\omega t}. \end{aligned} \tag{9}$$

Now, by virtue of this formula, the applied electromotive force E_1 consists of two exponential forces, one varying as $e^{i\omega t}$ and the other as $e^{-i\omega t}$. Hence it is easy to see that the currents are made up of two components, thus

$$I_j=J_j'e^{i\omega t}+J_j''e^{-i\omega t} \quad (j=1,2\ ..\ n) \tag{10}$$

and we have merely to use the exponential solution given above, substituting for λ,$i\omega$ and $-i\omega$ respectively. That is,

$$J_j'=\frac{1}{2}\frac{F'}{Z_{j1}(i\omega)} \text{ and } J_j''=\frac{1}{2}\frac{F''}{Z_{j1}(-i\omega)}$$

or

$$I_j=\frac{1}{2}\frac{Fe^{-i\theta}}{Z_{j1}(i\omega)}e^{i\omega t}+\frac{1}{2}\frac{Fe^{i\theta}}{Z_{j1}(-i\omega)}e^{-i\omega t}.$$

The second term is the conjugate imaginary of the first, so that

$$\begin{aligned} I_j&=R\frac{Fe^{-i\theta}}{Z_{j1}(i\omega)}e^{i\omega t}\\ &=R\frac{F}{Z_{j1}(i\omega)}e^{i(\omega t-\theta)}\\ &=R\frac{F}{|Z_{j1}(i\omega)|}e^{i(\omega t-\theta-\phi)}\\ &=\frac{F}{|Z(i\omega)|}\cos(\omega t-\theta-\phi). \end{aligned}$$

We thus arrive at the rule for the steady state solution:

If the applied e.m.f. is $F \cos(\omega t-\theta)$, substitute $i\omega$ for d/dt in the differential equations, determine the impedance function

$$Z(i\omega)=D(i\omega)/M(i\omega) \tag{11}$$

by the solution of the algebraic equations, and write it in the form

$$Z(i\omega)=|Z(i\omega)|e^{i\phi}. \tag{12}$$

Then the required solution is

$$I=\frac{F}{|Z(i\omega)|}\cos(\omega t-\theta-\phi). \tag{13}$$

This in compact form contains the whole theory of the symbolic solution of alternating current problems.

The Complementary Solution.—So far in the solutions which we have discussed the currents are of the same type as the impressed forces: that is to say in physical language, the currents are "forced" currents and vary with time in precisely the same manner as do the electromotive forces. Such currents are, however, in general only part of the total currents. In addition to the forced currents we have also the characteristic oscillations; or, in mathematical language, the complete solution must include both particular and complementary solutions. This may be shown as follows: Let $I_1', \ldots I_n'$ be solutions of the complementary equations,

$$\begin{aligned} &\left(L_{11}\frac{d}{dt}+R_{11}+\frac{1}{C_{11}}\int dt\right)I_1'+\cdot\cdot+\left(L_{1n}\frac{d}{dt}+R_{1n}+\frac{1}{C_{1n}}\int dt\right)I_n'=0,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &\left(L_{n1}\frac{d}{dt}+R_{n1}+\frac{1}{C_{n1}}\int dt\right)I_1'+\cdot\cdot+\left(L_{nn}\frac{d}{dt}+R_{nn}+\frac{1}{C_{nn}}\int dt\right)I_n'=0. \end{aligned} \tag{14}$$

Then if $I_1 \ldots I_n$ is a solution of (1), $I_1+I_1', \ldots I_n+I_n'$, is also a solution.

To derive the solution of the complementary system of equations (14), assume that a solution exists of the form

$$I_j'=J_j'\,e^{\lambda t} \qquad (j=1,2\ \ldots\ n)$$

so that $d/dt=\lambda$ and $\int dt=1/\lambda$. Substitute in equations (14) and cancel out the common factor $e^{\lambda t}$. Then we have

$$\begin{aligned} &z_{11}(\lambda)J_j'+\ \cdots\ +z_{1n}(\lambda)J_n'=0,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &z_{n1}(\lambda)J_1'+\ \cdots\ +z_{nn}(\lambda)J_n'=0. \end{aligned} \tag{15}$$

This is a system of n homogeneous equations in the unknown quantities $J_1', \ldots J_n'$. The condition that a finite solution

shall exist is that, in accordance with a well known principle of the theory of equations, the determinant of the coefficients shall vanish. That is,

$$D(\lambda)=\begin{vmatrix} z_{11}(\lambda) & \cdots & z_{1n}(\lambda) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ z_{n1}(\lambda) & \cdots & z_{nn}(\lambda) \end{vmatrix}=0. \tag{16}$$

Consequently the possible values of λ must be such that this equation is satisfied. In other words, λ must be a root of the equation $D(\lambda)=0$. Let these roots be denoted by $\lambda_1, \lambda_2, \ . \ . \ \lambda_m$. Then, assigning to λ any one of these values, we can determine the ratio J_j'/J_k' from any $(n-1)$ of the equations. That is to say, if we take

$$I_1'=C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}+ \ \cdots +C_me^{\lambda_m t}, \tag{17}$$

substitution in any $(n-1)$ of the equations determines $I_2', \ . \ . \ I_n'$. The m constants $C_1, \ . \ . \ C_m$ are so far, however, entirely arbitrary, and are at our disposal to satisfy imposed *boundary conditions.*

This introduces us to the idea of boundary conditions which is of the greatest importance in circuit theory. In physical language the boundary conditions denote the state of the system when the electromotive force is applied or when any change in the circuit constants occurs. The number of independent boundary conditions which can, in general, be satisfied is equal to the number of roots of the equation $D(\lambda)=0$. Evidently, therefore, it is physically impossible to impose more boundary conditions than this. On the other hand, if this number of boundary conditions is not specified, the complete solution is indeterminate: that is to say, the problem is not correctly set. As an example of boundary conditions, we may specify that the electromotive force is applied at time $t=0$, and that at this time all the currents in the inductances and all the charges on the condensers are zero.

So far we have been following the classical theory of linear differential equations. We have seen that the forced exponential solution and the derived steady state solution are extremely simple and are mere matters of elementary algebra. The practical difficulties in the classical method of solution begin with the determination of the constants $C_1, \ . \ . \ C_m$ of the complementary solution as well as the roots $\lambda_1, \ . \ . \ \lambda_m$ of the equation $D(\lambda)=0$. It is at this point that Heaviside broke

with classical methods, and by considering special boundary conditions of great physical importance, and particular types of impressed forces, laid the foundations of original and powerful methods of solution. We shall therefore at this point follow Heaviside's example and attack the problem from a different standpoint. In doing this we shall not at once take up an exposition of Heaviside's own method of attack. We shall first establish some fundamental theorems which are extremely powerful and will serve us as a guide in interpreting and rationalizing the Heaviside operational calculus.

CHAPTER II

THE SOLUTION WHEN AN ARBITRARY FORCE IS APPLIED TO THE NETWORK IN A STATE OF EQUILIBRIUM

In engineering applications of electric circuit theory there are four outstanding problems:

(1) The steady state distribution of currents and potentials when the network is energized by a sinusoidal electromotive force. This problem is the subject of the theory of alternating currents which forms the basis of our calculations of power lines and the more elaborate networks of communication systems.

(2) The distribution of currents and potentials in the network in response to an arbitrary electromotive force applied to the network in a state of equilibrium, i.e., applied when the currents and charges in the network are identically zero.

(3) The effect on the distribution of currents and potentials of suddenly changing a circuit constant or connection, such as opening or closing a switch, while the system is energized.

(4) The free oscillations of the network from an arbitary initial configuration. This problem will not be considered in detail for several reasons. In the first place it is of minor technical importance. Furthermore, if the initial state is producible by electromotive forces applied at accessible terminals of the network, the problem is reducible to and can be dealt with as a special case of problem (3). Finally, by reason of the arbitrary initial boundary conditions, the formulation and solution of the problem is inherently incapable of reduction to the elegant simplicity of those problems directly amenable to the operational calculus. The formal solution of problem (4) is, however, straightforward and is dealt with in treatises on dynamics. After evaluating the roots of the equation $D(\lambda)=0$ (see last paragraph of Chap. I), the constants of integration $C_1, \ . \ . \ C_m$ of the complementary solution must be determined to satisfy the initial conditions or configuration. Heaviside, by aid of his conjugate Energy Theorem, has given an elegant and rela-

tively simple expression for these constants in terms of the initial currents in the inductances and the initial charges on the condensers of the network.[1]

We shall base our further analysis of circuit theory on the solution of problem (2), for the following reasons:

(A) It is essentially a generalization of the Heaviside problem and its solution will furnish us a key to the correct understanding and interpretation of operational methods and lead to an auxiliary formula from which the rules of the operational calculus are directly deducible.

(B) The solution of problem (2) carries with it the solution of problem (3) and also serves as a basis for the theory of alternating currents.

(C) The solution of problem (2) leads directly to an extension of circuit theory to the case where the network contains variable elements: i.e., circuit elements which vary with time and in which non-linear relations obtain.

Problem (2) is therefore the fundamental problem of circuit theory and the formula which we shall now derive may be termed the fundamental formula of circuit theory.

Consider a network in any branch of which, say branch 1, a unit e.m.f. is inserted at time $t = 0$, the network having been previously in equilibrium. By unit e.m.f. is meant an electromotive force which has the value unity for all positive values of time ($t>0$). Let the resultant current in any branch, say branch n, be denoted by $A_{n1}(t)$. $A_{n1}(t)$ will be termed the *indicial admittance* of branch n with respect to branch 1—or, more fully, the transfer indicial admittance.

The indicial admittance, aside from its direct physical significance, plays a fundamental rôle in the mathematical theory of electric circuits. In words, it may be defined as follows: The indicial admittance, $A_{n1}(t)$, is equal to the ratio of the current in branch n, expressed as a time function, to the magnitude of the steady e.m.f. suddenly inserted at time $t=0$ in branch 1. It is evidently a function which is zero for negative values of time and approaches either zero or a steady value (the d.c. admit-

[1] The reader who wishes to study this problem further is referred to the following papers:

K. W. Wagner (E. N. T., Heft 11, Band 2, 1925).

M. S. Vallarta (A. I. E. E., Apr., 1926, "Heaviside's Proof of his Expansion Theorem").

tance) for all actual dissipative systems, as t approaches infinity. It may be noted that, aside from its mathematical determination, which will engage our attention later, it is an experimentally determinable function.

We note, in passing, an important property of the indicial admittance $A_{jk}(t)$, which is deducible from the reciprocal theorem:[1] this is that $A_{jk}(t)=A_{kj}(t)$. That is to say, the value of the transfer indicial admittance is unchanged by an interchange of the driving point and receiving point. It is therefore immaterial in the expression $A_{jk}(t)$ whether the e.m.f. is inserted in branch j and the current measured in branch k, or vice-versa. In general, unless we are concerned with particular branches, the subscripts will be omitted and we shall simply write $A(t)$, it being understood that any two branches or a single branch (for the case of equal subscripts) may be under consideration.

From the linear character of the network, it is evident that if a steady e.m.f. $E=E_\tau$ is inserted at time $t=\tau$, the network being in equilibrium, the resultant current is

$$E_\tau A(t-\tau).$$

Generalizing still further, suppose that steady e.m.fs. E_0, E_1, E_2, . . . E_n are impressed in the same branch at the respective times 0, τ_1, τ_2 . . . τ_n; the resultant current is evidently

$$E_0 A(t)+E_1 A(t-\tau_1)+\cdots+E_n A(t-\tau_n)=\sum_{j=0}^{n} E_j A(t-\tau_j). \quad (18)$$

To apply the foregoing to our problem we suppose that there is applied to the network, initially in a state of equilibrium, an e.m.f. $E(t)$ which has the following properties.

1. It is identically zero for $t<0$.
2. It has the value $E(0)$ for $0<t<\Delta t$.
3. It has the value $E(0)+\Delta_1 E$ for $\Delta t<t<2\Delta t$.
4. It has the value $E(0)+\Delta_1 E+\Delta_2 E$ for $2\Delta t<t<3\Delta t$.

. . . .

In other words it has the increment $\Delta_j E$ at time $t=j\Delta t$.

Evidently then the resultant current $I(t)$ is

$$E_0 A(t)+\Delta_1 E A(t-\Delta t)+\cdots+\Delta_n E.A(t-n\Delta t).$$

[1] Exceptions to this relation exist where the network contains sources of energy such as amplifiers. These need not engage our attention here.

Now evidently if the interval Δt is made shorter and shorter, then in the limit $\Delta t \to dt$ and $j\Delta t = \tau$ and

$$\Delta_j E = \frac{d}{d\tau} E(\tau) d\tau.$$

Passing to the limit in the usual manner this summation becomes a definite integral and we get

$$I(t) = E(0)A(t) + \int_0^t A(t-\tau) \frac{d}{d\tau} E(\tau) d\tau. \tag{19}$$

Finally by obvious transformations of the expression we arrive at the fundamental formula of circuit theory[1]

$$I(t) = \frac{d}{dt} \int_0^t A(t-\tau) E(\tau) d\tau, \tag{20}$$

$$= \frac{d}{dt} \int_0^t E(t-\tau) A(\tau) d\tau. \tag{20a}$$

For completeness we write down the following equivalents[2] of (20) and (20a)

$$I(t) = A(0)E(t) + \int_0^t A'(t-\tau)E(\tau) d\tau, \tag{20b}$$

$$= A(0)E(t) + \int_0^t A'(\tau)E(t-\tau) d\tau, \tag{20c}$$

$$= E(0)A(t) + \int_0^t E'(t-\tau)A(\tau) d\tau, \tag{20d}$$

$$= E(0)A(t) + \int_0^t E'(\tau)A(t-\tau) d\tau. \tag{20e}$$

where the primes denote differentiation with respect to the argument. Thus $A'(t) = d/dtA(t)$.

These equations are the fundamental formulas which mathematically relate the current to the type of applied electromotive force and the constants and connections of the system, and

[1] This important theorem, independently derived and published by the author, is actually the equivalent of a much older theorem in dynamics due to Duhamel. Substantially the same proof of the theorem as given in the text was independently communicated to the author by H. W. Nichols and by Stuart Ballantine.

[2] In case either $A(t)$ or $E(t)$ has discontinuities for positive values of t, we integrate through these discontinuities in the usual manner. When this is borne in mind, equations (20b) to (20e) are not restricted to continuous functions.

constitute the first part of the solution of our problem. The most important immediate deductions from these formulas are expressed in the following theorems.

1. The indicial admittance of an electrical network completely determines, within a single integration, the behavior of the network to all types of applied electromotive forces. As a corollary, a knowledge of the indicial admittance is the sole information necessary to completely predict the performance and characteristics of the system, including the steady state.

2. The applied e.m.f. and the indicial admittance are similarly and coequally related to the resultant current in the network. As a corollary the form of the current may be modified either by changing the constants and connections of the network or by modifying the form of the applied e.m.f.

3. Since the applied e.m.f. may be discontinuous these formulas determine not only the building up of the current in response to an applied e.m.f. but also its subsidence to equilibrium when the e.m.f. is removed and the network left to itself. In brief, formulas (20) reduce the whole problem to a determination of the indicial admittance of the network. In addition, as we shall see, they lead directly to an integral equation which determines this function.

It is of interest to show the relation between formulas (20) and the usual steady state equations. To do this let the e.m.f., applied at time $t=0$, be $E \sin(\omega t+\theta)$. Substitution in formula (20*c*) and rearrangement gives

$$
\begin{aligned}
I(t) &= A(0)E\sin(\omega t+\theta) \\
&+E\sin(\omega t+\theta)\int_0^t \cos\omega\tau A'(\tau)d\tau \\
&-E\cos(\omega t+\theta)\int_0^t \sin\omega\tau A'(\tau)d\tau
\end{aligned}
\tag{21}
$$

where
$$A'(t)=\frac{d}{dt}A(t).$$

Now this can be resolved into two parts

$$
\begin{aligned}
&E\sin(\omega t+\theta)\left\{A(0)+\int_0^\infty \cos\omega\tau A'(\tau)d\tau\right\} \\
&-E\cos(\omega t+\theta)\left\{\int_0^\infty \sin\omega\tau A'(\tau)d\tau\right\}
\end{aligned}
\tag{22}
$$

which is the *final steady state*, and

$$\begin{aligned} &-E\sin(\omega t+\theta)\int_t^\infty \cos\omega\tau A'(\tau)d\tau \\ &+E\cos(\omega t+\theta)\int_t^\infty \sin\omega\tau A'(\tau)d\tau \end{aligned} \tag{23}$$

which is the *transient distortion*, which ultimately dies away for sufficiently large values of time.

To correlate the foregoing expressions for the steady state with the usual formulas we observe that if the symbolic impedance of the network at frequency $\omega/2\pi$ be denoted by $Z(i\omega)$, and if we write

$$\frac{1}{Z(i\omega)}=\alpha(\omega)+i\beta(\omega)$$

then the steady state current is

$$E\,[\alpha(\omega)\,.\,\sin(\omega t+\theta)+\beta(\omega)\,.\,\cos(\omega t+\theta)].$$

Comparison with (22) gives at once

$$\alpha(\omega)=A(0)+\int_0^\infty \cos\omega\tau A'(\tau)d\tau, \tag{24}$$

$$\beta(\omega)=-\int_0^\infty \sin\omega\tau A'(\tau)d\tau. \tag{25}$$

The Integral Equation for the Indicial Admittance.—So far we have tacitly assumed that the indicial admittance is known. As a matter of fact its determination constitutes the essential part of our problem. It is, in fact, the Heaviside problem, and its investigation, to which we now proceed, will lead us directly to the operational calculus.

Heaviside's method in investigating this problem was intuitive and "experimental." We, however, shall establish a very general integral equation from which we shall directly deduce his methods and extensions thereof.

Let us suppose that an e.m.f. e^{pt}, where p is either a positive real quantity or complex with real part positive, is suddenly impressed on the network at time $t=0$. It follows from the foregoing theory that the resultant current $I(t)$ will be made up of two parts, (1) a forced exponential part which varies with time as e^{pt}, and (2) a complementary part which we shall denote by $y(t)$. The exponential or "forced" component is simply $e^{pt}/Z(p)$, where $Z(p)$ is functionally of the same form as the usual symbolic or complex

impedance $Z(i\omega)$. It is gotten from the differential equations of the problem, as explained in a preceding chapter, by replacing d^n/dt^n by p^n, cancelling out the common factor e^{pt}, and solving the resulting algebraic equations. The complementary or characteristic component, denoted by $y(t)$, depends on the constants and connections of the network, and on the value of p. It does not, however, contain the factor e^{pt} and it dies away for sufficiently large values of t, in all actual dissipative systems. Thus

$$I(t)=\frac{e^{pt}}{Z(p)}+y(t). \tag{26}$$

Now return to formula (20a) and replace $E(t)$ by e^{pt}. We get

$$I(t)=\frac{d}{dt}e^{pt}\int_0^t A(\tau)e^{-p\tau}\,d\tau$$

which can be written as

$$\frac{d}{dt}\left\{e^{pt}\int_0^\infty A(\tau)e^{-p\tau}\,d\tau-e^{pt}\int_t^\infty A(\tau)e^{-p\tau}\,d\tau\right\}.$$

Carrying out the indicated differentiation this becomes

$$I(t)=pe^{pt}\int_0^\infty A(\tau)e^{-p\tau}\,d\tau-pe^{pt}\int_t^\infty A(\tau)e^{-p\tau}\,d\tau+A(t). \tag{27}$$

Equating the two expressions (26) and (27) for $I(t)$ and dividing through by e^{pt} we get

$$\frac{1}{Z(p)}+y(t)e^{-pt}=p\int_0 A(\tau)e^{-p\tau}\,d\tau-p\int_t^\infty A(\tau)e^{-p\tau}\,d\tau+A(t)e^{-pt}. \tag{28}$$

This equation is valid for all values of t. Consequently if we set $t=\infty$ and if the real part of p is positive, only the first term on the right and the left hand side of the equation remain, the rest vanishes, and we get

$$\frac{1}{pZ(p)}=\int_0^\infty A(t)e^{-pt}\,dt. \tag{29}$$

This is an integral equation[1] *valid for all positive real values of p, which completely determines the indicial admittance $A(t)$.* It is

[1] An integral equation is one in which the unknown function appears under the sign of integration. (29) is an integral equation of the Laplace type. If $Z(p)$ is specified, $A(t)$ is uniquely determined. Methods for solving the integral equations are considered in detail later, in connection with the exposition of the operational calculus. The phrase "all positive values of p" will be understood as meaning all values of p in the right half of the complex plane.

on this equation that we shall base our discussion of operational methods and from which we shall derive the rules of the operational calculus. Equations (20) and (29) constitute a complete mathematical formulation of our problem, and from them the complete solution is obtainable without further recourse to the differential equations, or further consideration of boundary conditions.

CHAPTER III

THE HEAVISIDE PROBLEM AND THE OPERATIONAL EQUATION

By virtue of the analysis and formulas of the preceding chapters, we are now in a position to take up a systematic exposition and development of the Heaviside operational calculus. This analysis may be reviewed and summarized as follows:

The determination of the current in a network in response to an electromotive force $E(t)$, impressed on the network at reference time $t=0$ has been reduced to the mathematical solution of two equations: first the integral equation

$$\frac{1}{pZ(p)}=\int_0^\infty A(t)e^{-pt}dt \tag{29}$$

and second, the definite integral

$$I(t)=\frac{d}{dt}\int_0^t A(t-\tau)E(\tau)d\tau. \tag{20}$$

In equation (29), $Z(p)$ is the *impedance function* of the network and the characteristics, i.e., constants and connections, of the network enter into the problem only through $Z(p)$, and, therefore, $Z(p)$ completely specifies the network. If p is replaced by $i\omega$, where $\omega/2\pi$ is the frequency, then $Z(i\omega)$ is the complex steady state or symbolic impedance of alternating current theory. Thus, if $Z(p)$ is specified, equations (20) and (29) constitute a complete mathematical formulation of the problem, and their solution automatically takes care of the equilibrium boundary conditions. That is to say, contrary to the classical method of solution, the explicit determination of integration constants is completely dispensed with.

It will be observed that in deducing these equations we have merely postulated (1) the linear and invariable character of the network and (2) the existence of an exponential solution of the type $e^{pt}/Z(p)$ for positive values of p. Consequently, while we have so far discussed these formulas in terms of the determination of the current in a finite network, they are not limited in their

application to this specfic problem. In this connection it may be well to call attention explicitly to the following points.

The formulas and methods deduced above apply not only to finite networks, involving a finite system of linear equations, but to infinite networks and to transmission lines, involving infinite systems of equations, and partial differential equations: in fact to all electrical and dynamical systems in which the connections and constants are linear and invariable.

Secondly, the variable determined by formulas (20) and (29) need not, of course, be the current. It may equally well be the charge, potential drop, or any of the variables with which we may happen to be concerned. This fact may be explicitly recognized by writing the formulas as:

$$\frac{1}{pH(p)} = \int_0^\infty h(t)e^{-pt}\,dt, \tag{30}$$

$$x(t) = \frac{d}{dt}\int_0^t h(t-\tau)E(\tau)d\tau. \tag{31}$$

Here $E(t)$ is the applied e.m.f., $x(t)$ is the variable which we desire to determine (charge, current, potential drop, etc.), and

$$x = E/H(p) \tag{32}$$

is the operational equation. $H(p)$ therefore corresponds to and is determined in precisely the same way as the impedance $Z(p)$, but it may not have the physical significance or the dimensions of an impedance. Similarly, in character and function, $h(t)$ corresponds to the indicial admittance, though it may not have the same physical significance. It is a generalization of the indicial admittance and may be appropriately termed the *Heaviside Function*. Similarly $H(p)$ may be termed the *generalized impedance function.*

The Operational Equation.—The physical problem which Heaviside attacked and which led to his operational calculus was the determination of the response of a network or electrical system to a "unit e.m.f." (zero before, unity after time $t=0$) with, of course, the understanding that the system is in equilibrium when the electromotive force is applied. His problem is therefore, essentially that of the determination of the indicial admittance. In our exposition and critique of Heaviside's method of dealing with this problem we shall accompany an account of his own method of solution with a parallel solution from the corresponding integral equation of the problem.

Heaviside's first step in attacking this problem was to start with the differential equations, and replace the differential operator d/dt by the symbol p, and the operation $\int dt$ by $1/p$, thus reducing the equations to an algebraic form. He then wrote the impressed e.m.f. as 1 (unity), thus limiting the validity of the equations to values of $t>0$. The formal solution of the algebraic equations is straightforward and will be written as

$$h=1/H(p) \tag{33}$$

where h is the "generalized indicial admittance," or Heaviside function (denoting current, charge, potential or any variable with which we are concerned) and $H(p)$ is the corresponding generalized impedance. Thus, if we are concerned with the current in any part of the network, we write

$$A=1/Z(p). \tag{34}$$

The more general notation is desirable, however, as indicating the wider applicability of the equation.

The equations

$$h=1/H(p)$$
$$A=1/Z(p)$$

are the *Heaviside operational equations.* They are, as yet, purely symbolic and we have still the problem of determining their explicit meaning and in particular the significance of the operator p.

Comparison of the Heaviside operational equations with the integral equations (29) and (30) of the preceding chapter leads to the following fundamental theorem.

The Heaviside operational equations

$$A=1/Z(p)$$
$$h=1/H(p)$$

are merely the symbolic or short-hand equivalents of the corresponding integral equations

$$\frac{1}{pZ(p)}=\int_0^\infty A(t)e^{-pt}\,dt$$
$$\frac{1}{pH(p)}=\int_0^\infty h(t)e^{-pt}\,dt.$$

The integral equations, therefore, supply us with the meaning and significance of the operational equations, and from them the rules of the operational calculus are deducible.

By virtue of this theorem, we have the advantage, at the outset, of a key to the meaning of Heaviside's operational equations, and a means of checking and deducing his rules of solution. This will serve us as a guide throughout our further study.

Returning now to Heaviside's own point of view and method of attack, his reasoning may be described somewhat as follows:— The operational equation

$$h = 1/H(p)$$

is the full equivalent of the differential equations of the problem and must therefore contain the information necessary to the solution provided we can determine the significance of the symbolic operator p. The only way of doing this, when starting with the operational equation, is one of induction: that is, we must compare the operational equation with known solutions of specific problems and thus attempt to infer by induction general rules for interpreting the operational equation and converting it into the required explicit solution.

The Power Series Solution

Let us start with the simplest possible problem: the current in response to a "unit e.m.f." in a circuit consisting of an inductance L in series with a resistance R.

The differential equation of the problem is

$$L\frac{d}{dt}A + RA = 1, \qquad t>0,$$

where A is the indicial admittance. Consequently replacing d/dt by p, the operational equation is

$$A = \frac{1}{pL+R}.$$

The explicit solution is easily derived: it is

$$A = \frac{1}{R}(1-e^{-\alpha t})$$

where $\alpha = R/L$. Note that this makes the current initially zero, so that the equilibrium boundary condition at $t=0$ is satisfied.

Now suppose that we expand the operational equation in inverse powers of p: we get, formally,

$$A = \frac{1}{pL}\,\frac{1}{1+\alpha/p} = \frac{1}{R}\,\frac{\alpha}{p}\,\frac{1}{1+\alpha/p} = \frac{1}{R}\left\{\frac{\alpha}{p} - \left(\frac{\alpha}{p}\right)^2 + \left(\frac{\alpha}{p}\right)^3 - \left(\frac{\alpha}{p}\right)^4 + \cdots\right\}$$

by the Binomial Theorem.

Now expand the explicit solution as a power series in t: it is

$$A = \frac{1}{R}\left\{\frac{\alpha t}{1!} - \frac{(\alpha t)^2}{2!} + \frac{(\alpha t)^3}{3!} - \dots \right\}.$$

Comparing the two expansions we see at once that the operational expansion is converted into the explicit solution by assigning to the symbol $1/p^n$ the value $t^n/n!$. It was from this kind of inductive inference that Heaviside arrived at his power series solution.

Now there are several important features in the foregoing which require comment. In the first place the operational equation is converted into the explicit solution only by a particular kind of expansion, namely an expansion in inverse powers of the operator p. For example, if in the operational equation

$$A = \frac{1}{R}\frac{\alpha/p}{1+\alpha/p}$$

we replace $1/p$ by $t/1!$ we get

$$A = \frac{1}{R}\frac{\alpha t}{1+\alpha t}$$

which is incorrect. Furthermore, if we expand in ascending instead of descending powers of p, namely

$$A = \frac{1}{R}\left\{1 - (p/\alpha) + (p/\alpha)^2 - \dots\dots \right\}$$

no correlation with the explicit solution is possible and no significance can be attached to the expansion. We thus infer the general principle, and we shall find this inference to be correct, that the operational equation is convertible into the explicit solution only by the proper choice of expansion of the impedance function, or rather its reciprocal.

In the second place we notice that in writing down the operational equation and then converting it into the explicit solution no consideration has been given to the question of boundary conditions. This is one of the great advantages of the operational method: the boundary conditions, *provided they are those of equilibrium*, are automatically taken care of. This will be illustrated in the next example:

Let a "unit e.m.f." be impressed on a circuit consisting of resistance R, inductance L, and capacity C: required the resultant charge on the condenser.

The differential equation for the charge Q is

$$\left(L\frac{d^2}{dt^2}+R\frac{d}{dt}+1/C\right)Q=1, \qquad t>0.$$

Consequently the operational formula is

$$Q=\frac{1}{Lp^2+Rp+1/C}$$
$$=\frac{1}{Lp^2}\frac{1}{1+a/p+b/p^2} \text{ where } a=\frac{R}{L} \text{ and } b=\frac{1}{LC}.$$

This can be expanded by the Binomial Theorem as

$$Q=\frac{1}{Lp^2}\left\{1-\left(\frac{a}{p}+\frac{b}{p^2}\right)+\left(\frac{a}{p}+\frac{b}{p^2}\right)^2-\left(\frac{a}{p}+\frac{b}{p^2}\right)^3+\cdots\right\}.$$

Performing the indicated operations and collecting in inverse powers of p, the first few terms of the expansion are:—

$$\frac{1}{Lp^2}\left\{1-\frac{c_1}{p}-\frac{c_2}{p^2}+\frac{c_3}{p^3}+\frac{c_4}{p^4}-\frac{c_5}{p^5}-\frac{c_6}{p^6}+\cdots\right\}$$

where

$$c_1=a$$
$$c_2=b-a^2$$
$$c_3=2ab-a^3$$
$$c_4=b^2-3a^2b+a^4$$
$$c_5=3ab^2-4a^3b+a^5$$
$$c_6=b^3-6a^2b^2+5a^4b-a^6$$
$$\cdots\cdots\cdots\cdots\cdots$$

We infer therefore that in accordance with the rule of replacing $1/p^n$ by $t^n/n!$ the solution is:—

$$Q=\frac{1}{L}\left\{\frac{t^2}{2!}-c_1\frac{t^3}{3!}-c_2\frac{t^4}{4!}+c_3\frac{t^5}{5!}+c_4\frac{t^6}{6!}-\cdots\right\}.$$

Owing to the complicated character of the coefficients in the expansion, the series cannot be recognized and summed by inspection. If, however, we put $R=0$ then $a=0$, and the series becomes

$$C\left\{\frac{1}{2!}\left(\frac{t}{\sqrt{LC}}\right)^2-\frac{1}{4!}\left(\frac{t}{\sqrt{LC}}\right)^4+\frac{1}{6!}\left(\frac{t}{\sqrt{LC}}\right)^6-\cdots\right\}$$

whence

$$Q=C\{1-\cos(t/\sqrt{LC})\}.$$

We have still to verify this solution by comparison with the explicit solution of the differential equation. This is of the form

$$Q=C+k_1e^{\lambda_1 t}+k_2e^{\lambda_2 t}$$

where k_1 and k_2 are constants which must be chosen to satisfy the boundary conditions and λ_1, λ_2 are the roots of the equation

$$L\lambda^2+R\lambda+1/C=0.$$

Now since we have two arbitrary constants we satisfy the equilibrium conditions by making Q and dQ/dt zero at $t=0$. whence

$$C+k_1+k_2=0,$$
$$\lambda_1 k_1+\lambda_2 k_2=0,$$

and

$$k_1=\lambda_2 C/(\lambda_1-\lambda_2),$$
$$k_2=\lambda_1 C/(\lambda_2-\lambda_1).$$

We have also

$$\lambda_1=-\frac{a}{2}+\sqrt{\left(\frac{a}{2}\right)^2-b},$$

$$\lambda_2=-\frac{a}{2}-\sqrt{\left(\frac{a}{2}\right)^2-b}.$$

Writing down the power series expansion of

$$Q=C+k_1e^{\lambda_1 t}+k_2e^{\lambda_2 t},$$

then

$$Q=(C+k_1+k_2)+(k_1\lambda_1+k_2\lambda_2)\frac{t}{1!}$$
$$+(k_1\lambda_1^2+k_2\lambda_2^2)\frac{t^2}{2!}+\cdots.$$

Introducing the values of k_1, k_2, λ_1, λ_2 given above and comparing with the power series derived from the operational solution we see that they are identical term by term.

This example illustrates two facts. First the power series expansions may be complicated, laborious to derive and of such form that they cannot be recognized and summed by inspection. In fact in arbitrary networks of a large number of meshes or degrees of freedom the evaluation of the coefficients of the power series expansion is extremely laborious.

On the other hand, in such cases, the solution by the classical method presents difficulties far more formidable—in fact insuperable difficulties from a practical standpoint. First there is the location of the roots of the function $H(\lambda)$, which in arbitrary networks is a practical impossibility without a prohibitive amount of labor. Secondly there is the determination of the integration constants to satisfy the imposed boundary conditions: a process,

which, while theoretically straightforward, is actually in practice extremely laborious and complicated. We note these points in passing; a more complete estimate of the value of the power series solution will be made later.

To summarize the preceding: Heaviside, generalizing from specific examples otherwise solvable, arrived at the following rule:—

Expand the right hand side of the operational equation

$$h=1/H(p)$$

in inverse powers of p: thus

$$h \sim a_0+a_1/p+a_2/p^2+\cdots+a_n/p^n+\cdots$$

and then replace $\frac{1}{p^n}$ by $t^n/n!$. *The operational equation is thereby converted into the explicit power series solution:—*

$$h=a_0+a_1 t/1!+a_2 t^2/2!+\cdots+a_n t^n/n!+\cdots \tag{35}$$

As stated above, this rule was arrived at by pure induction and generalization from the known solution of specific problems. It cannot, therefore, theoretically be regarded as satisfactorily established. The rule can, however, be directly deduced from the integral equation

$$\frac{1}{pH(p)}=\int_0^\infty h(t)e^{-pt}\,dt.$$

To its derivation from this equation we shall now proceed.

First suppose we *assume* that $h(t)$ admits of the power series expansion

$$h_0+h_1 t/1!+h_2 t^2/2!+\cdots\cdots$$

Substitute this assumed expansion in the integral, and integrate term by term. The right hand side of the integral equation becomes formally

$$h_0/p+h_1/p^2+h_2/p^3+\cdots\cdots$$

by virtue of the formula

$$\int_0^\infty \frac{t^n}{n!}e^{-pt}=\frac{1}{p^{n+1}} \text{ for } p>0.$$

Now expand the left hand side of the integral equation asymptotically in inverse process of p: it becomes

$$a_0/p+a_1/p^2+a_2/p^3+\cdots\cdots$$

where

$$a_0+a_1/p+a_2/p^2+\cdots\cdot$$

is the asymptotic expansion of $1/H(p)$. Comparing the two expansions and making a term by term identification, we see that $h_n = a_n$ and

$$h(t) = a_0 + a_1 t/1! + a_2 t^2/2! + \cdots$$

which agrees with the Heaviside formula.

This procedure, however, while giving the correct result has serious defects from a mathematical point of view. For example, the asymptotic expansion of $1/H(p)$ has usually only a limited region of convergence, and it is only in this region that term by term integration is legitimate. Furthermore we have *assumed* the possibility of expanding $h(t)$ in a power series: an assumption to which there are serious theoretical objections, and which, furthermore, is not always justified. A more satisfactory derivation, and one which establishes the condition for the existence of a power series expansion, proceeds as follows:—

Let $1/H(p)$ be a function which admits of the formal asymptotic expansion

$$\sum_0^\infty a_n/p^n$$

and let it include no component which is asymptotically representable by a series all of whose terms are zero, that is a function $\phi(p)$ such that the limit, as $p \to \infty$, of $p^n \phi(p)$ is zero for every value of n. Such a function is e^{-p}. With this restriction understood, start with the integral equation, and integrate by parts: we get

$$\frac{1}{H(p)} = h(0) + \int_0^\infty e^{-pt} h^{(1)}(t)dt$$

where $h^{(n)}(t)$ denotes $d^n/dt^n\, h(t)$. Now let p approach infinity: in the limit the integral vanishes and by virtue of the asymptotic expansion

$$1/H(p) \sim \sum_0^\infty a_n/p^n, \tag{36}$$

$1/H(p)$ approaches the limit a_0. Consequently

$$h(0) = a_0.$$

Now integrate again by parts: we get

$$p(1/H(p) - a_0) = h^{(1)}(0) + \int_0^\infty e^{-pt} h^{(2)}(t)dt.$$

Again let p approach infinity: in the limit the left hand side of the equation becomes a_1 and we have

$$h^{(1)}(0)=a_1.$$

Proceeding by successive partial integrations we thus establish the general relation

$$h^{(n)}(0)=a_n.$$

But by Taylor's theorem, the power series expansion of $h(t)$ is simply

$$h(t)=h(0)+h^{(1)}(0)t/1!+h^{(2)}(0)t^2/2!+\cdot\cdot\cdot\cdot\cdot$$

whence, *assuming the convergence of this expansion*, we get

$$h(t)=a_0+a_1t/1!+a_2t^2/2!+\ \cdots\ =\sum_0^\infty a_n t^n/n! \qquad (35)$$

which establishes the power series solution. It should be carefully noted, however, that it does not establish the convergence of the power series solution. As a matter of fact, however, I know of no physical problem in which $H(p)$ satisfies the conditions for an asymptotic expansion, where the power series solution is not convergent. On the other hand many physical problems exist, including those relating to transmission lines, where a power series solution is not derivable and does not exist.

The process of expanding the operational equation in such a form as to permit of its being converted into the explicit solution is what Heaviside calls "algebrizing" the equation. In the case of the power series solution the process of algebrizing consists in expanding the reciprocal of the impedance function in an asymptotic series,[1] thus

$$1/H(p)\sim a_0+a_1/p+a_2/p^2+\cdot\cdot\cdot\cdot\cdot\cdot$$

Regarded as an expansion in the variable p, instead of as a purely symbolic expansion, this series has usually only a limited region of convergence. This fact need not bother us, however, as the series we are really concerned with is

$$a_0+a_1t/1!+a_2t^2/2!+\ \cdot\ \cdot\ \cdot\ \cdot$$

[1] The symbol $\sim$, frequently employed in the following pages, is to be understood as signifying "is asymptotically equal to." In other words the two sides of the equation are not, in general, identities but are *asymptotically* equivalent or equal.

It is interesting to note in passing that the latter series is what Borel, the French mathematician, calls the *associated function* of the former, and is extensively employed by him in his researches on the summability of divergent series.

The process of "algebrizing," as in the examples discussed above, may often be effected by a straightforward binomial expansion. In other cases the form of the generalized impedance function $H(p)$ will indicate by inspection the appropriate procedure. A general process, applicable in all cases where a power series exists, is as follows. Write

$$1/H(p)=1/H\left(\frac{1}{x}\right)=G(x). \tag{36}$$

Now expand $G(x)$ as a Taylor's series: thus formally

$$G(x)=G(0)+G^{(1)}(0)\frac{x}{1!}+G^{(2)}(0)\frac{x^2}{2!}+\cdots$$

where

$$G^{(n)}(0)=\left[\frac{d^n}{dx^n}G(x)\right]_{x=0}. \tag{37}$$

Denote $\dfrac{G^{(n)}(0)}{n!}$ by a_n, replace x^n by $1/p^n$, and we have

$$G(x) = 1/H(p) = a_0+a_1/p+a_2/p^2+\cdots$$

This process of "algebrizing" is formally straightforward and always possible. As implied above, however, in many problems much shorter modes of expansion suggest themselves from the form of the function $H(p)$.

We note here, in passing, that the necessary and sufficient conditions for the existence of a power series solution is the possibility of the formal expansion of $G(x)$ as a power series in x.

At this point a brief critical estimate of the scope and value of the power series solution may be in order. As stated above, in a certain important class of problems relating to transmission lines, a power series does not exist, though a closely related series in fractional powers of t may often be derived. Consequently the power series solution is of restricted applicability. Where, however, a power series does exist, in directness and simplicity of derivation it is superior to any other form of solution. Its chief defect, and a very serious defect indeed, is that except where the power series can be recognized and summed, it is usually practically useless for computation and interpretation except for relatively small values of the time t. This

disadvantage is inherent and attaches to all power series solutions. For this reason I think Heaviside overestimated the value of power series as practical or working solutions, and that some of his strictures against orthodox mathematicians and their solutions may be justly urged against the power series solution. He was quite right in insisting that a solution must be capable of either interpretation or computation and quite right in ridiculing those formal solutions which actually conceal rather than reveal the significance of the original differential equations of the problem. On the other hand, the following remark of his indicates to me that Heaviside had a quite exaggerated idea of the value and fundamental character of power series in general: "I regret that the result should be so complicated. But the only alternatives are other equivalent infinite series, or else a definite integral which is of no use until it is evaluated, when the result must be the series (135), or an equivalent one." As a matter of fact the properties of most of the important functions of mathematical physics have been investigated and their values computed by methods other than series expansions. I may add that in technical work the power series solution has proved to be of restricted utility, while definite integrals, which Heaviside[1] particularly despised, have proved quite useful.

The Expansion Theorem Solution[2]

We pass now to the consideration of another extremely important form of solution. Heaviside gives this solution without proof: we shall therefore merely state the solution and then derive it from the integral equation.

Given the operational equation

$$h = 1/H(p)$$

which has the significance discussed above: i.e., the response of the network to a "unit e.m.f." The explicit solution may be written as

$$h = \frac{1}{H(0)} + \sum_{1}^{n} \frac{e^{p_k t}}{p_k H'(p_k)} \tag{38}$$

[1] Vide a remark of his to the effect that some mathematicians took refuge in a definite integral and called that a solution.

[2] This terminology is due to Heaviside. A more appropriate and physically significant expression would be "The Solution in terms of normal or characteristic vibrations."

where $p_1, p_2 \ . \ . \ . \ p_n$ are the n roots of the equation

$$H(p)=0$$

and

$$H'(p_k)=\left[\frac{d}{dp}H(p)\right]_{p=p_k}. \tag{39}$$

As remarked above, this solution, referred to by him as *The Expansion Theorem*, was stated by Heaviside without proof; how he arrived at it will probably always remain a matter of conjecture.[1] Its derivation from the integral equation is, however, a relatively simple matter, though in special cases troublesome questions arise.

The derivation of the expansion solution from the integral equation

$$\frac{1}{pH(p)}=\int_0^\infty h(t)e^{-pt}\,dt$$

follows immediately from the partial fraction expansion

$$\frac{1}{pH(p)}=\frac{1}{pH(0)}+\sum_{j=1}^{n}\frac{1}{(p-p_j)p_jH'(p_j)} \tag{40}$$

where $p_1, p_2 \ . \ . \ . \ p_n$ are the roots of the equation $H(p)=0$, and

$$H'(p_j)=\left[\frac{d}{dp}H(p)\right]_{p=p_j}. \tag{41}$$

Partial fraction expansions of this type are fully discussed in treatises on algebra and the calculus and the conditions for their existence established. Before discussing the restrictions imposed on $H(p)$ by this expansion, we shall first, assuming its existence, derive the expansion theorem solution.

By virtue of (40) the integral equation is

$$\frac{1}{pH(0)}+\sum_{1}^{n}\frac{1}{(p-p_j)p_jH'(p_j)}=\int_0^\infty h(t)e^{-pt}\,dt. \tag{42}$$

The expansion on the left hand side suggests a corresponding expansion on the right hand side: that is, we suppose that

$$h(t)=h_0(t)+h_1(t)+h_2(t)+\cdots+h_n(t) \tag{43}$$

[1] Since this was written it has been independently shown by K. W. Wagner and M. S. Vallarta (see footnote to p. 14) that a proof based on his "conjugate theorem" can be reconstructed from scattered papers of Heaviside.

and specify that these component functions shall satisfy the equations

$$\frac{1}{pH(0)} = \int_0^\infty h_0(t)e^{-pt}\,dt \tag{44}$$

$$\frac{1}{(p-p_j)p_j H'(p_j)} = \int_0^\infty h_j(t)e^{-pt}\,dt \qquad j=1,2\ldots n. \tag{45}$$

It follows at once from (43) and direct addition of equations (44) and (45) that (42) is satisfied and hence is solved provided $h_0, \ldots h_n$ can be evaluated from (44) and (45).

Now since

$$\int_0^\infty e^{\lambda t}e^{-pt}\,dt = \frac{1}{p-\lambda} \tag{46}$$

provided the real part of λ is not positive (a condition satisfied in all network problems), we see at once that equations (42) and (43) are satisfied by taking

$$h_0(t) = h_0 = \frac{1}{H(0)}, \tag{47}$$

$$h_j(t) = \frac{e^{p_j t}}{p_j H'(p_j)}, \qquad j=1,2,\ldots n.$$

Consequently from (43) and (47) it follows that

$$h(t) = \frac{1}{H(0)} + \sum_1^n \frac{e^{p_j t}}{p_j H'(p_j)} \tag{48}$$

which establishes the expansion theorem solution.

As implied above, the partial fraction expansion (40), on which the expansion theorem solution depends, imposes certain restrictions on the impedance function $H(p)$. Among these are that $H(p)$ must have no zero root, no repeated roots, and $1/H(p)$ must be a proper fraction. In all finite networks these conditions are satisfied, or by a slight modification, the operational equation can be reduced to the required form. The case of repeated roots, which may occur where the network involves a unilateral source of energy such as an amplifier, can be dealt with by assuming unequal roots and then letting the roots approach equality as a limit. Without entering upon these questions in detail, however, we can very simply and directly establish the proposition that the expansion theorem gives the solution whenever a solution in terms of normal or characteristic vibrations exists. The proof of this proposition proceeds as follows.

It is known from the elementary theory of linear differential equations that the general solution of the set of differential equations, of which the operational equation is $h=1/H(p)$, is of the form

$$h(t)=C_0+\sum_1^n C_j e^{p_j t}$$

where p_j is the jth root of $H(p)=0$, and C_0, C_1 . . . . C_n are constants of integration which must be so chosen as to satisfy the system of differential equations and the imposed boundary conditions. The summation is extended over all the roots of $H(p)$ which is supposed not to have a zero root or repeated roots.

Now substitute this known form of solution in the integral equation of the problem and carry out the integration term by term. We get

$$\frac{1}{H(p)}=C_0+p\sum\frac{C_j}{p-p_j}. \tag{49}$$

Setting $p=0$, we have at once

$$C_0=1/H(0). \tag{50}$$

To determine C_j let $p=p_j+q$ where q is a small quantity ultimately to be set equal to zero, and write the equation as

$$C_0H(p)+\sum\frac{pH(p)}{p-p_j}C_j=1. \tag{51}$$

If now $p=p_j+q$ and q approaches zero, this becomes in the limit

$$p_jH'(p_j)C_j=1 \tag{52}$$

or

$$C_j=\frac{1}{p_jH'(p_j)}, \tag{53}$$

whence
$$h(t)=\frac{1}{H(0)}+\sum\frac{e^{p_jt}}{p_jH'(p_j)} \tag{54}$$

which is the Expansion Theorem Solution.

We shall not attempt to discuss here cases where the expansion solution breaks down though such cases exist. In every such case, however, the breakdown is due to the failure of the impedance function $H(p)$ to satisfy the conditions necessary for the partial fraction expansion (40), and correlatively the non-existence of a solution in normal vibrations. Furthermore, it is

usually possible by simple modifications to deduce a modified expansion solution. It may be added here, that while the proof given above is also limited implicitly to finite networks, the expansion solution is valid in most transmission line problems.

Let us now illustrate how the expansion solution works by applying it to a few simple examples. Take first the case considered in the preceding chapter in connection with the power series solution. Required the charge Q on a condenser C in series with an inductance L and resistance R in rseponse to a "unit e.m.f." The operational equation is

$$Q=\frac{1}{Lp^2+Rp+1/C}$$

or

$$Q=\frac{1}{L}\frac{1}{p^2+2\alpha p+\omega^2}$$

where $\alpha = R/2L$ and $\omega^2 = 1/LC$.

The roots of the equation $H(p)=0$ are the roots of the equation

$$p^2+2\alpha p+\omega^2=0$$

whence

$$p_1=-\alpha+\sqrt{\alpha^2-\omega^2}=-\alpha+\beta,$$
$$p_2=-\alpha-\sqrt{\alpha^2-\omega^2}=-\alpha-\beta.$$

Also $H'(p)=2L(p+\alpha)$, so that

$$H'(p_1)=2\beta L$$
$$H'(p_2)=-2\beta L$$

and

$$1/H(0)=1/L\omega^2=C.$$

Inserting these expressions in the Expansion Theorem Solution (38), we get

$$Q=C-\frac{e^{-\alpha t}}{2\beta L}\left(\frac{e^{\beta t}}{\alpha-\beta}-\frac{e^{-\beta t}}{\alpha+\beta}\right).$$

It is now easy to verify the fact that this solution satisfies the differential equations and the boundary condition

$$Q=0 \text{ and } dQ/dt=0 \text{ at time } t=0.$$

If $\omega>\alpha$, β is a pure imaginary

$$\beta=i\omega\sqrt{1-(\alpha/\omega)^2}=i\omega'$$

and

$$Q=C-\frac{e^{-\alpha t}}{\omega' L}\frac{\omega'\cos\omega' t+\alpha\sin\omega' t}{\alpha^2+\omega'^2}.$$

In connection with this problem we note two advantages of the expansion solution, as compared with the power series solution: (1) it is much simpler to derive from the operational equation, and (2) its numerical computation is enormously easier. A table of exponential and trigometric functions enables us to evaluate Q for any value of t almost at once whereas in the case of the power series solution the labor of computation for large values of t is very great. A third and very important advantage of the expansion solution in this particular problem is that without detailed computation we can deduce by mere inspection the general character of the function and the effect of the circuit parameters on its form: an advantage which never attaches to the power series solution.

This last property of the particular solution above is extremely important. The ideal form of solution, particularly in technical problems, is one which permits us to infer the general character and properties of the function and the effect of the circuit constants on its form, without detailed computation. A solution which possesses these properties, even if its exact computation is not possible without prohibitive labor, is far superior to a solution which, while completely computable, tells us nothing without detailed computation. It is for this reason that some of the derived forms of solution, discussed later, are of such importance. In fact a solution which requires detailed computation before it yields the information implied in it is merely equivalent to an experimentally determined solution.

Unfortunately the advantages attaching to the expansion solution of the specific problem just discussed, do not, in general, characterize the expansion solution. The following disadvantages should be noted. First, the location of the roots of the impedance function $H(p)$ is practically impossible in the case of arbitrary networks of more than a few degrees of freedom. In the second place, when the number of degrees of freedom is large it is not only impossible to deduce the significance of the solution by inspection, but the computation becomes extremely laborious. In such cases, the practical value of the expansion solution depends, just as in the power series solution, on the possibility of recognizing and summing the expansion. This will be clear in the case of transmission lines, where the roots of $H(p)$ are infinite in number and the direct computation of the expansion solution (except in the case of the non-inductive cable) is quite impossible.

CHAPTER IV

SOME GENERAL FORMULAS AND THEOREMS FOR THE SOLUTION OF OPERATIONAL EQUATIONS

We have seen that the operational equation

$$h=1/H(p)$$

is the symbolic or short-hand equivalent of the integral equation

$$\frac{1}{pH(p)}=\int_0^\infty h(t)e^{-pt}\,dt$$

and from the latter we have deduced two very important forms of the Heaviside solution. In recognizing the equivalence of these two equations we have a very great advantage and are able, in fact, to base the operational calculus on deductive instead of inductive reasoning. In this chapter we shall employ this equivalence to establish certain general formulas and theorems for the solution of operational equations. That is to say, we shall make use of the principles that (1) any method applicable to the solution of the integral equation supplies us with a corresponding method for the solution of the operational equation, and (2) a solution of any specific integral equation gives at once the solution of the corresponding operational equation. We turn therefore to a brief discussion of the appropriate methods for solving the integral equation.

It has been noted above that any solution of the integral equation supplies a solution of the corresponding operational equation. This principle enables us to take advantage of the fact that a very large number of infinite integrals of the type

$$\int_0^\infty f(t)e^{-pt}\,dt$$

have been evaluated. *The evaluation of every infinite integral of this type supplies us, therefore, with the solution of an operational equation.*

Of course, not all the operational equations so solvable have physical significance. Many however, do. Below is a list of infinite integrals with their known solutions, accompanied by the corresponding operational equation and its explicit solution.

All of these solutions are directly applicable to important technical problems. It may be remarked in passing that the infinite integrals have for the most part been evaluated by advanced mathematical methods which need not concern us here.

Table of Infinite Integrals, the Corresponding Operational Equations, and Their Explicit Solutions

(a) $$\int_0^\infty e^{-pt}e^{-\lambda t}\,dt=\frac{1}{p+\lambda},$$
$$h=\frac{p}{p+\lambda}=e^{-\lambda t}.$$

(b) $$\int_0^\infty e^{-pt}\frac{t^n}{n!}\,dt=1/p^{n+1},$$
$$h=\frac{1}{p^n}=t^n/n!.$$

(c) $$\int_0^\infty e^{-pt}\frac{1}{\sqrt{\pi t}}\,dt=\frac{1}{\sqrt{p}},$$
$$h=\sqrt{p}=1/\sqrt{\pi t}.$$

(d) $$\int_0^\infty e^{-pt}\frac{(2t)^n}{1.3.5\cdots(2n-1)}\frac{dt}{\sqrt{\pi t}}=\frac{1}{p^n\sqrt{p}},$$
$$h=\frac{\sqrt{p}}{p^n}=\frac{(2t)^n}{1.3.5\cdots(2n-1)}\frac{1}{\sqrt{\pi t}}.$$

(e) $$\int_0^\infty e^{-pt}\frac{t^n}{n!}e^{-\lambda t}\,dt=\frac{1}{(p+\lambda)^{n+1}},$$
$$h=\frac{p}{(p+\lambda)^{n+1}}=\frac{t^n}{n!}e^{-\lambda t}.$$

(f) $$\int_0^\infty e^{-pt}\sqrt{\frac{\lambda}{\pi}}\,\frac{e^{-\lambda/t}}{t\sqrt{t}}\,dt=e^{-2\sqrt{\lambda p}},$$
$$h=pe^{-2\sqrt{\lambda p}}=\sqrt{\frac{\lambda}{\pi}}\,\frac{e^{-\lambda/t}}{t\sqrt{t}}.$$

(g) $$\int_0^\infty e^{-pt}\frac{e^{-\lambda/t}}{\sqrt{\pi t}}\,dt=\frac{e^{-2\sqrt{\lambda p}}}{\sqrt{p}},$$
$$h=\sqrt{p}\,e^{-2\sqrt{\lambda p}}=\frac{e^{-\lambda/t}}{\sqrt{\pi t}}.$$

(h) $$\int_0^\infty e^{-pt}\sin\lambda t\,dt=\frac{\lambda}{p^2+\lambda^2},$$
$$h=\frac{p\lambda}{p^2+\lambda^2}=\sin\lambda t.$$

(i) $$\int_0^\infty e^{-pt}\cos\lambda t\,dt=\frac{p}{p^2+\lambda^2},$$

$$h=\frac{p^2}{p^2+\lambda^2}=\cos\lambda t.$$

(j) $$\int_0^\infty e^{-pt}e^{-\mu t}\cos\lambda t\,dt=\frac{p+\mu}{(p+\mu)^2+\lambda^2},$$

$$h=\frac{p^2+\mu p}{(p+\mu)^2+\lambda^2}=e^{-\mu t}\cos\lambda t.$$

(k) $$\int_0^\infty e^{-pt}e^{-\mu t}\sin\lambda t\,dt=\frac{\lambda}{(p+\mu)^2+\lambda^2},$$

$$h=\frac{p\lambda}{(p+\mu)^2+\lambda^2}=e^{-\mu t}\sin\lambda t.$$

(l) $$\int_0^\infty e^{-pt}J_0(\lambda t)dt=\frac{1}{\sqrt{p^2+\lambda^2}},$$

$$h=\frac{p}{\sqrt{p^2+\lambda^2}}=J_0(\lambda t).$$

(m) $$\int_\lambda^\infty e^{-pt}J_0(\sqrt{t^2-\lambda^2})dt=\frac{e^{-\lambda\sqrt{p^2+1}}}{\sqrt{p^2+1}},$$

$$h=\frac{p}{\sqrt{p^2+1}}e^{-\lambda\sqrt{p^2+1}}=0 \text{ for } t<\lambda$$

$$=J_0(\sqrt{t^2-\lambda^2}) \text{ for } t>\lambda.$$

(n) $$\int_0^\infty e^{-pt}J_n(\lambda t)dt=\frac{1}{r}\left(\frac{r-p}{\lambda}\right)^n, \quad r^2=p^2+\lambda^2,$$

$$h=\frac{p}{r}\left(\frac{r-p}{\lambda}\right)^n=J_n(\lambda t).$$

(p) $$\int_0^\infty e^{-pt}e^{-\lambda t}I_0(\lambda t)dt=\frac{1}{\sqrt{p^2+2\lambda p}},$$

$$h=\frac{1}{\sqrt{1+2\lambda/p}}=e^{-\lambda t}I_0(\lambda t).$$

In formulas (l), (m), (n), $J_n(x)$ denotes the Bessel function of order n and argument x. In formula (p), $I_0(x)$ denotes the Bessel function $J_0(ix)$ where $i=\sqrt{-1}$.

This list might be greatly extended. As it is, we are in possession of a set of solutions of operational equations which occur in important technical problems and which will be employed later.

The foregoing emphasizes the practical and theoretical importance of recognizing the equivalence of the integral and operational equations. With this equivalence in mind, the solution of an

operational equation is often reduced to a mere reference to a table of infinite integrals. Heaviside did not recognize this equivalence. As a consequence many of his solutions of transmission line problems are extremely laborious and involved and in the end unsatisfactory because expressed in involved power series.

Not all the infinite integrals corresponding to the operational equations of physical problems have been evaluated or can be recognized without transformation. This statement corresponds exactly with the fact that a table of integrals is not always sufficient but must be supplemented by general methods of integration. We turn, therefore, to stating and discussing some general theorems applicable to the solution of operational equations.

In the derivation of the operational theorems, which constitute the general rules of the operational calculus, the following proposition, due to Borel and known as Borel's theorem, will be frequently employed.[1]

If the functions $f(t)$, $f_1(t)$, and $f_2(t)$ are defined by the integral equations

$$F(p) = \int_0^\infty f(t)e^{-pt}\,dt$$

$$F_1(p) = \int_0^\infty f_1(t)e^{-pt}\,dt$$

$$F_2(p) = \int_0^\infty f_2(t)e^{-pt}\,dt$$

and if the functions F, F_1 and F_2 satisfy the relation

$$F(p) = F_1(p).F_2(p)$$

then

$$f(t) = \int_0^t f_1(\tau)f_2(t-\tau)\,d\tau$$
$$= \int_0^t f_2(\tau)f_1(t-\tau)\,d\tau.$$

The operational theorems will now be stated and briefly proved from the integral equation identity.

[1] For a proof of this important theorem the reader is referred to Borel, "Lecons sur les Séries Divergentes" (1901), p. 104; to Bromwich, "Theory of Infinite Series," pp. 280–281; or to Ford, "Studies on Divergent Series and Summability," pp. 93–94 (being Vol. II of the Michigan University Science Series, published by Macmillan). The proof depends on Jacobi's transformation of a double integral: see Edward's "Integral Calculus," 1922, Vol. II, pp. 14–15.

Theorem I

If in the operational equation

$$h = 1/H(p)$$

the generalized impedance function $H(p)$ can be expanded in a sum of terms, thus

$$\frac{1}{H(p)} = \frac{1}{H_1(p)} + \frac{1}{H_2(p)} + \cdots + \frac{1}{H_n(p)},$$

and if the auxiliary operational equations

$$h_1 = \frac{1}{H_1(p)}$$

$$h_2 = \frac{1}{H_2(p)}$$

$$\cdots\cdots$$

can be solved, then

$$h = h_1 + h_2 + \cdots + h_n.$$

This theorem is too obvious to require detailed proof: in fact it is self-evident. The power series and expansion theorem solutions are examples of its application. In general, however, the appropriate form of expansion of $1/H(p)$ will depend on the particular problem in hand. The theorem, as it stands is a formal statement of the fact that solutions can often be obtained by an appropriate expansion whereas the equation cannot be solved as it stands.

Theorem II

If $h = h(t)$ and $g = g(t)$ are defined by the operational equations

$$h = 1/H(p)$$

$$g = 1/pH(p)$$

then

$$g(t) = \int_0^t h(\tau)d\tau.$$

To prove this theorem we start with the integral equations

$$\frac{1}{pH(p)} = \int_0^\infty h(t)e^{-pt}\,dt,$$

$$\frac{1}{p^2H(p)} = \int_0^\infty g(t)e^{-pt}\,dt.$$

The second of these is in form for an immediate application of Borel's theorem since

$$\frac{1}{p^2H(p)} = \frac{1}{p} \cdot \frac{1}{pH(p)}.$$

The functions f_1 and f_2 of Borel's theorem then satisfy the equations

$$\frac{1}{p} = \int_0^\infty f_1(t)e^{-pt}\,dt,$$
$$\frac{1}{pH(p)} = \int_0^\infty f_2(t)e^{-pt}\,dt.$$

It follows at once that

$$f_1(t) = 1$$
$$f_2(t) = h(t)$$

whence by Borel's theorem

$$g(t) = \int_0^t h(\tau)d\tau.$$

THEOREM III

If $h = h(t)$ and $g = g(t)$ are defined by the operational equations

$$h = 1/H(p)$$
$$g = p/H(p)$$

then

$$g(t) = \frac{d}{dt}h(t)$$

provided $h(0) = 0$.

The integral equations of the problem are

$$\frac{1}{pH(p)} = \int_0^\infty h(t)e^{-pt}\,dt,$$
$$\frac{1}{H(p)} = \int_0^\infty g(t)e^{-pt}\,dt.$$

Integrating the first of these by parts we have,

$$\frac{1}{pH(p)} = \frac{1}{p}h(0) + \frac{1}{p}\int_0^\infty h'(t)e^{-pt}\,dt$$

where $h'(t) = d/dt\ h(t)$.

If $h(0) = 0$, we have at once

$$\frac{1}{H(p)} = \int_0^\infty h'(t)e^{-pt}\,dt.$$

Comparison with the integral equation for $g(t)$ shows at once that $g(t) = h'(t)$, since the integral equation determines the function uniquely.

Theorems II and III establish the characteristic Heaviside operations of replacing $1/p$ by $\int_0^t dt$ and p by d/dt.

Theorem IV

If in the operational equation

$$h = 1/H(p)$$

the generalized impedance function can be factored in the form

$$H(p) = H_1(p) \cdot H_2(p)$$

and if the auxiliary operational equations

$$h_1 = 1/H_1(p)$$
$$h_2 = 1/H_2(p)$$

define the auxiliary variables h_1 *and* h_2, *then*

$$h(t) = \frac{d}{dt}\int_0^t h_1(\tau)h_2(t-\tau)d\tau$$
$$= \frac{d}{dt}\int_0^t h_2(\tau)h_1(t-\tau)d\tau.$$

This theorem is immediately deducible from Borel's theorem and theorems II and III, as follows.

The integral equations are

$$\frac{1}{pH(p)} = p\frac{1}{pH_1(p)} \cdot \frac{1}{pH_2(p)} = \int_0^\infty h(t)e^{-pt}\,dt$$
$$\frac{1}{pH_1(p)} = \int_0^\infty h_1(t)e^{-pt}\,dt$$
$$\frac{1}{pH_2(p)} = \int_0^\infty h_2(t)e^{-pt}\,dt.$$

Now define an auxiliary function $g(t)$ by the operational equation

$$g = \frac{1}{pH(p)}.$$

Then

$$\frac{1}{pH_1(p)} \cdot \frac{1}{pH_2(p)} = \int_0^\infty g(t)e^{-pt}\,dt$$

and by Borel's theorem

$$g(t) = \int_0^t h_1(\tau)h_2(t-\tau)d\tau$$
$$= \int_0^t h_2(\tau)h_1(t-\tau)d\tau.$$

From this equation it follows that $g(0) = 0$, and hence comparing the operational equations for h and g, we have by aid of Theorem III

$$h(t) = \frac{d}{dt}g(t)$$

and hence

$$h(t)=\frac{d}{dt}\int_0^t h_1(\tau)h_2(t-\tau)d\tau$$
$$=\frac{d}{dt}\int_0^t h_2(\tau)h_1(t-\tau)d\tau.$$

This theorem is extremely important, although not stated or employed by Heaviside himself. We shall make use of it in establishing two important general theorems and shall have frequent occasion to employ it in specific problems occurring in connection with the subsequent discussion of transmission theory.

THEOREM V

If $h=h(t)$ and $g=g(t)$ are defined by the operational equations

$$h=\frac{1}{H(p)}$$
$$g=\frac{1}{H(p+\lambda)}$$

where λ is a positive real parameter, then

$$g(t)=(1+\lambda\int_0^t dt)e^{-\lambda t}h(t).$$

To prove this theorem we start with the integral equations

$$\frac{1}{pH(p)}=\int_0^\infty h(t)e^{-pt}\,dt$$
$$\frac{1}{pH(p+\lambda)}=\int_0^\infty g(t)e^{-pt}\,dt.$$

In the first of these equations replace the symbol p by $q+\lambda$: we get

$$\frac{1}{q+\lambda}\cdot\frac{1}{H(q+\lambda)}=\int_0^\infty h(t)e^{-\lambda t}e^{-qt}\,dt$$

and then to preserve our original notation replace the symbol q by p, whence

$$\frac{1}{(p+\lambda)H(p+\lambda)}=\int_0^\infty h(t)e^{-\lambda t}\,e^{-pt}\,dt. \qquad \text{(a)}$$

The integral equation in $g(t)$ can be written as

$$\left(1+\frac{\lambda}{p}\right)\frac{1}{(p+\lambda)H(p+\lambda)}=\int_0^\infty g(t)e^{-pt}\,dt. \qquad \text{(b)}$$

Comparing equations (a) and (b) it follows at once from theorems I and II that

$$g(t)=\left(1+\lambda\int_0^t dt\right)h(t)e^{-\lambda t}.$$

From the foregoing, the following auxiliary theorem is immediately deducible.

THEOREM Va

If $h=h(t)$ and $g=g(t)$ are defined by the operational equations

$$h=\frac{1}{H(p)}$$

$$g=\frac{p}{(p+\lambda)H(p+\lambda)}$$

then

$$g(t)=h(t)e^{-\lambda t}.$$

The proof of this theorem will be left as an exercise to the reader.

THEOREM VI

If $h=h(t)$ and $g=g(t)$ are defined by the operational equations

$$h=1/H(p)$$
$$g=1/H(\lambda p)$$

where λ is a positive real parameter, then

$$g(t)=h(t/\lambda).$$

We start with the integral equations

$$\frac{1}{pH(p)}=\int_0^\infty h(t)e^{-pt}\,dt$$

$$\frac{1}{pH(\lambda p)}=\int_0^\infty g(t)e^{-pt}\,dt$$

and in the first of these equations we replace p by λq and t by τ/λ, whence it becomes

$$\frac{1}{qH(\lambda q)}=\int_0^\infty h\left(\frac{\tau}{\lambda}\right)e^{-q\tau}\,d\tau.$$

Now replacing the symbols q and τ by p and t respectively, we have

$$\frac{1}{pH(\lambda p)}=\int_0^\infty h(t/\lambda)e^{-pt}\,dt$$

whence by comparison with the integral equation in $g(t)$ it follows at once that

$$g(t)=h(t/\lambda).$$

This theorem is often useful in making a convenient change in the time scale and eliminating superfluous constants.

Theorem VII

If $h = h(t)$ and $g = g(t)$ are defined by the operational equations

$$h = \frac{1}{H(p)}$$

$$g = \frac{e^{-\lambda p}}{H(p)}$$

where λ is a positive real quantity, then

$$\begin{aligned} g(t) &= 0 \quad \text{for } t < \lambda \\ &= h(t-\lambda) \quad \text{for } t > \lambda. \end{aligned}$$

This is a very important theorem in connection with transmission line problems where retardation, due to finite velocity of propagation, occurs. Its proof proceeds as follows:

If the auxiliary function $k = k(t)$ is defined by the operational equation

$$k = e^{-\lambda p}$$

then by Theorem IV,

$$g(t) = \frac{d}{dt}\int_0^t k(\tau)h(t-\tau)d\tau. \tag{a}$$

Now, corresponding to the operational equation $k = e^{-\lambda p}$ we have the integral equation

$$\frac{e^{-\lambda p}}{p} = \int_0^\infty k(t)e^{-pt}\,dt.$$

The solution of this integral equation, which is easily verified by direct substitution in the infinite integral, is

$$\begin{aligned} k(t) &= 0 \quad \text{for } t < \lambda \\ &= 1 \quad \text{for } t > \lambda. \end{aligned}$$

Hence equation (a) becomes

$$\begin{aligned} g(t) &= 0 \quad \text{for } t < \lambda \\ &= \frac{d}{dt}\int_\lambda^t h(t-\tau)d\tau \quad \text{for } t > \lambda \\ &= h(t-\lambda) \quad \text{for } t > \lambda. \end{aligned}$$

Theorem IV, employed in the preceding proof, as stated above, is extremely important and we shall have frequent occasion to employ it in specific problems. We shall now apply it to deduce an important theorem which extends the operational calculus

to arbitrary impressed forces, whereas heretofore the operational equation $h=1/H(p)$ applied only to the case of a "unit e.m.f." impressed on the system.

It will be recalled from a previous chapter that if $x(t)$ denotes the response of a network to an arbitrary force $f(t)$, impressed at time $t=0$, and if $h(t)$ denotes the corresponding response to a "unit e.m.f.," then

$$x(t)=\frac{d}{dt}\int_0^t h(\tau)f(t-\tau)d\tau \tag{31}$$

and

$$\frac{1}{pH(p)}=\int_0^\infty h(t)e^{-pt}\,dt. \tag{30}$$

Now $f(t)$ may be of such form that the infinite integral

$$\int_0^\infty f(t)e^{-pt}\,dt$$

can be evaluated and has the value $F(p)/p$: thus

$$\int_0^\infty f(t)e^{-pt}\,dt=\frac{1}{p}F(p). \tag{55}$$

This is possible, of course, for many important types of applied forces, including the sinusoidal.

It follows at once from Theorem IV that $x(t)$ satisfies and is determined by the integral equation

$$\frac{1}{p}\frac{F(p)}{H(p)}=\int_0^\infty x(t)e^{-pt}\,dt. \tag{56}$$

We have thus succeeded, by virtue of Theorem IV in expressing the response of a network to an arbitrary e.m.f. impressed at time $t=0$, by an integral equation of the same form as that expressing the response to a "unit e.m.f." That is to say we have, at least formally, extended the operational calculus explicitly to the case of arbitrary impressed forces.

We now translate the foregoing into the corresponding operational theorem.

Theorem VIII

If the operational equation

$$h=1/H(p)$$

expresses the response of a network to a "unit e.m.f." and if an arbitrary e.m.f. E impressed at time $t=0$, is expressible by the operational equation

$$E=V(p)$$

or the infinite integral

$$\int_0^\infty E(t)e^{-pt}\,dt=\frac{V(p)}{p}$$

then the response x of the network to the arbitrary force is given by he operational equation

$$x=\frac{V(p)}{H(p)},$$

and $x(t)$ is determined by the integral equation

$$\frac{1}{p}\frac{V(p)}{H(p)}=\int_0^\infty x(t)e^{-pt}\,dt.$$

Theorem IX

If the operational equation

$$h=1/H(p)$$

is reducible to the form

$$h=\frac{F(p)}{1+\lambda K(p)}$$

where λ is a real parameter, and if the auxiliary functions $f=f(t)$ and $k=k(t)$ are defined by the auxiliary operational equations

$$f=F(p)$$
$$k=K(p)$$

then $h(t)$ is determined by the Poisson integral equation

$$h(t)=f(t)-\lambda\frac{d}{dt}\int_0^t h(\tau)k(t-\tau)d\tau.$$

This theorem is of considerable practical importance in connection with the approximate and numerical solutions of operational equations when the operational equation and the equivalent Laplace integral equation prove refractory. In such cases, as will be shown later, the numerical solution of the Poisson integral equations can often be rapidly and accurately effected, and in many cases the qualitative properties of $h(t)$ can be deduced from it without detailed numerical solution.

The proof of this theorem proceeds as follows:

By virtue of the relation $h=1/H(p)$ the operational equation

$$h=\frac{F(p)}{1+\lambda K(p)}$$

can be written as

$$h+\lambda\frac{K(p)}{H(p)}=F(p)$$

$$h=F(p)-\lambda\frac{K(p)}{H(p)}.$$

A direct application of Borel's theorem or Theorem IV gives at once the explicit equivalent

$$h(t)=f(t)-\lambda\frac{d}{dt}\int_0^t h(\tau)k(t-\tau)d\tau.$$

The preceding theorems, together with the power series and expansion theorem solutions formulate the most important rules of the operational calculus, and are constantly employed in the solution of electrotechnical problems. On the other hand, the table of infinite integrals furnishes the solution of a set of operational equations, which are of the greatest usefulness in the systematic study of propagation phenomena in transmission systems which will engage our attention. Before taking up this study, however, we shall first solve a few specific problems which will serve as an introduction to asymptotic and divergent solutions involving Heaviside's so-called "fractional differentiation."

Problem A: Current Entering the Non-Inductive Cable.—The non-inductive cable is a smooth line with distributed resistance R and capacity C per unit length; for the present we neglect inductance and leakage. A consideration of cable problems leads to some of the most interesting questions relating to operational methods, particularly to questions regarding divergent expansions. It would seem best to allow specific problems to serve as an introduction to these general questions.

The differential equations of the cable are

$$RI=-\frac{\partial}{\partial x}V$$

$$C\frac{d}{dt}V=-\frac{\partial}{\partial x}I \tag{57}$$

where x is the distance, measured along the cable from any fixed point, I is the current at point x, and V the corresponding potential.

Replacing d/dt by the operator p, we have

$$\left.\begin{aligned} RI &= -\frac{\partial}{\partial x}V \\ pCV &= -\frac{\partial}{\partial x}I. \end{aligned}\right\} \qquad (58)$$

Eliminating, successively, V and I from these equations, we get

$$pRC \cdot I = \frac{\partial^2}{\partial x^2}I$$

and

$$pRC \cdot V = \frac{\partial^2}{\partial x^2}V$$

These equations have the general solution

$$V = V_1 e^{-\gamma x} + V_2 e^{\gamma x} \qquad (59)$$

$$I = \sqrt{\frac{pC}{R}}\,[V_1 e^{-\gamma x} - V_2 e^{\gamma x}] \qquad (60)$$

where

$$\gamma = \sqrt{pRC}. \qquad (61)$$

The term in $e^{-\gamma x}$ represents the direct wave and the term in $e^{\gamma x}$ the reflected wave. V_1 and V_2 are constants which must be so chosen as to satisfy the imposed boundary conditions at the terminals of the cable.

For the present we shall assume that the line is infinitely long so that the reflected wave is absent. We shall also assume that a voltage E is impressed directly on the cable at $x = 0$: we have then,

$$V = Ee^{-x\sqrt{pCR}} = Ee^{-\sqrt{ap}} \qquad (62)$$

$$I = \sqrt{\frac{pC}{R}}\,Ee^{-x\sqrt{pCR}} = \sqrt{\frac{pC}{R}}\,Ee^{-\sqrt{ap}} \qquad (63)$$

where a denotes x^2RC.

To convert these to operational equations let us suppose that E is a "unit e.m.f." (zero before, unity after time $t = 0$). We have then, in operational notation

$$V = e^{-\sqrt{ap}} \qquad (64)$$

$$I = \sqrt{\frac{pC}{R}}\,e^{-\sqrt{ap}}. \qquad (65)$$

Now suppose that $x=0$ so that $a=0$, in other words consider a point at the cable terminals. Then

$$V=1$$
$$I=\sqrt{\frac{pC}{R}}. \tag{66}$$

The first of these equations means that V is simply the impressed voltage, zero before, unity after time $t=0$, as of course, it should be from physical considerations.

Corresponding to the operational equation

$$I=\sqrt{\frac{pC}{R}}. \tag{66}$$

we have the integral equation

$$\sqrt{\frac{C}{R}}\frac{1}{\sqrt{p}}=\int_0^\infty I(t)e^{-pt}\,dt. \tag{67}$$

The solution of this is known (see formula (c) of the preceding table of integrals): it is

$$I=\sqrt{\frac{C}{R}}\frac{1}{\sqrt{\pi t}}=\sqrt{\frac{C}{\pi Rt}}. \tag{68}$$

Heaviside arrived at this solution from considering the known solution of the same problem in the theory of heat flow. He therefore inferred that the operational equation

$$I=\sqrt{p}$$

has the explicit solution

$$I=1/\sqrt{\pi t}.$$

This is correct; we, however, have derived it directly from the integral equation of the problem and the known integral

$$\frac{1}{\sqrt{p}}=\int_0^\infty e^{-pt}\frac{dt}{\sqrt{\pi t}}. \tag{69}$$

We thus see from the foregoing that, if a "unit e.m.f." is impressed on the cable terminals, the current entering the cable is initially infinite and dies away in accordance with the formula $\sqrt{C/\pi Rt}$. The case is, of course, idealized and the infinite initial value of the current results from our ignoring the distributed inductance of the cable, which, no matter how small, keeps the initial current finite, as we shall see later.

Now let us go a step farther; suppose that in addition to distributed resistance R and capacity C, the cable also has

distributed leakage G per unit length. The differential equations are now

$$R\cdot I=-\frac{\partial}{\partial x}V$$
$$(Cp+G)V=-\frac{\partial}{\partial x}I. \tag{70}$$

Consequently it follows that in the operational equation for the current entering the cable we need only replace Cp by $Cp+G$. Therefore, when leakage is included, equation (66) is to be replaced by

$$I=\sqrt{\frac{pC+G}{R}}=\sqrt{\frac{C}{R}}\sqrt{p+\lambda} \tag{71}$$

where $\lambda=G/C$.

The corresponding integral equation is, of course,

$$\sqrt{\frac{C}{R}}\frac{\sqrt{p+\lambda}}{p}=\int_0^\infty I(t)e^{-pt}\,dt. \tag{72}$$

We shall give two solutions of this problem; first the solution of the integral equation, and second the typical Heaviside solution directly from the operational equation.

Equation (72) may be written as

$$\sqrt{\frac{C}{R}}\frac{(1+\lambda/p)}{\sqrt{p+\lambda}}=\int_0^\infty I(t)e^{-pt}\,dt. \tag{73}$$

Now suppose that $J(t)$ is the solution of the equation

$$\frac{1}{\sqrt{p+\lambda}}=\int^\infty J(t)e^{-pt}\,dt \tag{74}$$

it follows at once from Theorems (I) and (II) of the preceding chapter that

$$I(t)=\sqrt{\frac{C}{R}}\left(1+\lambda\int_0^t dt\right)J(t). \tag{75}$$

Also from formula (c) of the table of integrals and Theorem (Va) the solution of (74) is

$$J(t)=\frac{e^{-\lambda t}}{\sqrt{\pi t}} \tag{76}$$

whence

$$I(t)=\sqrt{\frac{C}{\pi R}}\left\{\frac{e^{-\lambda t}}{\sqrt{t}}+\lambda\int_0^t\frac{e^{-\lambda t}}{\sqrt{t}}dt\right\}. \tag{77}$$

The integral appearing in (77) can not be evaluated in finite terms; it is easily expressible as a series, however, by repeated integration by parts. Thus

$$\int_0^t \frac{e^{-\lambda t}}{\sqrt{t}} dt = 2\int_0^t e^{-\lambda t} d\sqrt{t} = 2\sqrt{t}\, e^{-\lambda t} + 2\lambda \int_0^t e^{-\lambda t}\sqrt{t}\, dt.$$

Proceeding in this way by repeated partial integration we get for the integral term of (77)

$$2\sqrt{t}\, e^{-\lambda t}\left\{1 + \frac{2\lambda t}{1.3} + \frac{(2\lambda t)^2}{1.3.5} + \cdots\right\}. \tag{78}$$

The straightforward Heaviside solution is obtained by expanding the operational equation as follows:

$$\begin{aligned} I &= \sqrt{\frac{C}{R}}\sqrt{p+\lambda} \\ &= \sqrt{\frac{C}{R}}\left(1+\frac{\lambda}{p}\right)^{1/2}\sqrt{p} \\ &= \sqrt{\frac{C}{R}}\left[1+\frac{1}{2}\frac{\lambda}{p} - \frac{1}{2.4}\left(\frac{\lambda}{p}\right)^2 + \frac{1.3}{2.4.6}\left(\frac{\lambda}{p}\right)^3 - \cdots\right]\sqrt{p}. \end{aligned}$$

Identifying $\sqrt{p}$ with $1/\sqrt{\pi t}$ (from known solutions of allied problems) and substituting for $1/p^n$ multiple integrations of the nth order we get

$$I = \sqrt{\frac{C}{\pi R t}}\left\{1 + \frac{(2\lambda t)}{2} - \frac{(2\lambda t)^2}{2.3.4} + \frac{1.3(2\lambda t)^3}{2.3.4.5.6} - \cdots\right\}. \tag{79}$$

It can be verified that this solution is convergent and equivalent to (77).

This problem, while simple and of minor technical interest, will serve to introduce us to the very important and interesting question of asymptotic series solutions.

An asymptotic series, for our purposes, may be defined as a series expansion of a function, which, while divergent, may be used for numerical computation, and which exhibits the behavior of the function for sufficiently large values of the argument.

Let us return to equation (77). We observe that the series solution (78) of the definite integral becomes increasingly laborious to compute as the value of t increases. This remark applies with even greater force to the Heaviside solution (79) on account of the alternating character of the series. Right here we have an excellent example of what I regard as Heaviside's exaggerated sense of the importance of series solutions as compared with definite integrals. Consider the solution in the form of (77) as

compared with Heaviside's series solution (79). The former is incomparably easier to interpret and to compute, either by numerical integration or by means of an integraph or planimeter. In fact the series (79) is practically unmanageable except for small values of t.

Returning to the question of an asymptotic expansion of the solution (77), we observe that the definite integral appearing in that equation can be written as,

$$\int_0^t \frac{e^{-\lambda t}}{\sqrt{t}}dt = \int_0^\infty \frac{e^{-\lambda t}}{\sqrt{t}}dt - \int_t^\infty \frac{e^{-\lambda t}}{\sqrt{t}}dt \tag{80}$$

provided λ is positive, as it is in this case. Now the value of the infinite integral is known; it is $\sqrt{\pi/\lambda}$. Consequently

$$\int_0^t \frac{e^{-\lambda t}}{\sqrt{t}}dt = \sqrt{\frac{\pi}{\lambda}} - \int_t^\infty \frac{e^{-\lambda t}}{\sqrt{t}}dt; \tag{81}$$

furthermore,

$$\int_t^\infty \frac{e^{-\lambda t}}{\sqrt{t}}dt = -\frac{1}{\lambda}\int_t^\infty \frac{1}{\sqrt{t}}de^{-\lambda t} = \frac{1}{\lambda}\frac{e^{-\lambda t}}{\sqrt{t}} - \frac{1}{2\lambda}\int_t^\infty \frac{e^{-\lambda t}}{t\sqrt{t}}dt.$$

Integrating again by parts we get

$$\frac{1}{\lambda}\frac{e^{-\lambda t}}{\sqrt{t}} - \frac{1}{2\lambda^2}\frac{e^{-\lambda t}}{t\sqrt{t}} + \frac{1.3}{2^2\lambda^2}\int_t^\infty \frac{e^{-\lambda t}}{t^2\sqrt{t}}dt.$$

Continuing this process, we get

$$\begin{aligned}\int_t^\infty \frac{e^{-\lambda t}}{\sqrt{t}}dt = \frac{e^{-\lambda t}}{\lambda\sqrt{t}}\Bigg[1 - \frac{1}{2\lambda t} + \frac{1.3}{(2\lambda t)^2} - \frac{1.3.5}{(2\lambda t)^3} \\ + \cdots + (-1)^n\frac{1.3.5\ldots(2n-1)}{(2\lambda t)^n}\Bigg] \\ -\frac{(-1)^n}{\lambda}\frac{1.3.5\ldots(2n+1)}{2(2\lambda)^n}\int_t^\infty \frac{e^{-\lambda t}}{t^{n+1}\sqrt{t}}dt.\end{aligned} \tag{82}$$

Now this series is divergent, that is, if we continue out far enough in the series the terms begin to increase in value without limit. On the other hand, if we stop with the nth term the error is represented by the integral term in (82) and this is *less than*

$$\frac{(-1)^n}{\lambda\sqrt{t}}\frac{1.3.5\ldots(2n-1)}{(2\lambda t)^n}e^{-\lambda t}. \tag{83}$$

Consequently *the error committed in stopping with any term in the series is less than the value of that term.* Therefore if we stop with the smallest term in the series, the error is less than the smallest term and decreases with increasing values of t.

We can therefore write the solution (77) as

$$I \sim \sqrt{\frac{\lambda C}{R}} + \sqrt{\frac{C}{\pi R t}} e^{-\lambda t} \left\{ \frac{1}{2\lambda t} - \frac{1.3}{(2\lambda t)^2} + \frac{1.3.5}{(2\lambda t)^3} - \cdots \right\}. \tag{84}$$

The first term, since $\lambda = G/C$, is simply $\sqrt{G/R}$, the d.c. admittance of the leaky cable. The divergent series shows how the current approaches this final steady value.

In this particular problem no asymptotic solution is derivable directly from the operational equation, at least by the straightforward Heaviside processes. Asymptotic solutions, however, constitute a large and important part of Heaviside's transmission line solutions. We shall therefore discuss next a problem for which Heaviside obtained both convergent and divergent series expansions.

Problem B: Terminal Voltage on Cable with "Unit E.M.F." Impressed on Cable through Condenser.—We now take up a problem for which Heaviside obtained a divergent solution, and which will introduce us to the theory of his divergent solutions and so-called "fractional differentiation." We suppose a "unit e.m.f." impressed on an infinitely long cable of distributed resistance R and capacity C per unit length through a condenser of capacity C_0: required the voltage V at the cable terminals. The operational equation of the problem is derived as follows:—

We know from the problem just discussed that the current entering the cable whose terminal voltage is V, is, in operational notation

$$\sqrt{\frac{Cp}{R}} V.$$

But the current flowing into the condenser is

$$C_0 p(1-V)$$

since the voltage across the condenser is $1-V$. Equating these two expressions we get

$$V = \frac{pC_0}{pC_0 + \sqrt{pC/R}} \tag{85}$$

which is the operational equation of the problem.

This may be written as

$$V = \frac{1}{1 + \frac{1}{C_0}\sqrt{\frac{C}{R}}\frac{1}{\sqrt{p}}}$$

$$= \frac{1}{1+\sqrt{a/p}},$$

where

$$\sqrt{a}=\frac{1}{C_0}\sqrt{C/R}.$$

Now expanding this by the binomial theorem

$$\begin{aligned}V&=1-\sqrt{\frac{a}{p}}+\frac{a}{p}-\frac{a}{p}\sqrt{\frac{a}{p}}+\left(\frac{a}{p}\right)^2-\cdot\cdot\\&=1+\frac{a}{p}+\left(\frac{a}{p}\right)^2+\cdot\cdot\cdot\cdot\\&-(1+\frac{a}{p}+\left(\frac{a}{p}\right)^2+\cdot\cdot)\sqrt{\frac{a}{p}},\\&=1+\frac{at}{1!}+\frac{(at)^2}{2!}+\cdot\cdot\cdot\cdot\\&-\left(\frac{2at}{1}+\frac{(2at)^2}{1.3.}+\frac{(2at)^3}{1.3.5.}+\cdot\cdot\right)\frac{1}{\sqrt{\pi at}}\end{aligned}\tag{86}$$

by the usual Heaviside rules of "algebrizing."

It is worth while verifying this from the integral equation of the problem. We have

$$\frac{1}{p}\frac{1}{1+\sqrt{a/p}}=\int_0^\infty V(t)e^{-pt}\,dt.\tag{87}$$

The left hand side can be written as

$$\frac{1}{p-a}-\frac{1}{p-a}\sqrt{\frac{a}{p}}$$

and by the formulas and theorems given in a preceding section the solution can be recognized at once as:—

$$V(t)=e^{at}-\sqrt{\frac{a}{\pi}}e^{at}\int_0^t\frac{e^{-a\tau}}{\sqrt{\tau}}d\tau.\tag{88}$$

This can also be written as

$$V(t)=\sqrt{\frac{a}{\pi}}e^{at}\int_t^\infty\frac{e^{-a\tau}}{\sqrt{\tau}}d\tau.\tag{89}$$

If the definite integral of (88) is evaluated by successive partial integrations it will be found in agreement with the Heaviside solution (86).

Now the solution (86) is in powers of t and while absolutely convergent becomes progressively more difficult to interpret and compute as the value of t increases. From (89), however, we can derive a divergent or asymptotic solution applicable both for interpretation and computation, when the value of t is suffi-

ciently large. As in the example discussed before, the asymptotic expansion results from repeated partial integrations; thus

$$\begin{aligned}\int_t^\infty \frac{e^{-a\tau}}{\sqrt{\tau}}\,d\tau &= -\frac{1}{a}\int_t^\infty \frac{1}{\sqrt{\tau}}\,d\,e^{-a\tau}\\ &= \frac{e^{-at}}{a\sqrt{t}}-\frac{1}{2a}\int_t^\infty \frac{e^{-a\tau}}{\tau\sqrt{\tau}}\,d\tau\\ &= \frac{e^{-at}}{a\sqrt{t}}+\frac{1}{2a^2}\int_t^\infty \frac{1}{\tau\sqrt{\tau}}\,d\,e^{-a\tau}\\ &= \frac{e^{-at}}{a\sqrt{t}}-\frac{e^{-at}}{2a^2t\sqrt{t}}+\frac{1.3}{2^2a^2}\int_t^\infty \frac{e^{-a\tau}}{\tau^2\sqrt{\tau}}\,d\tau\end{aligned}$$

and finally

$$\frac{e^{-at}}{a\sqrt{t}}\left\{1-\frac{1}{2at}+\frac{1.3}{(2at)^2}-\frac{1.3.5}{(2at)^3}+\cdots\right\}. \tag{90}$$

The series (90) is divergent just as is (82) of a preceding problem and the error committed by stopping with the smallest term, is of the same character and subject to the same discussion. With this understanding we write the solution (89) as

$$V(t)\sim\frac{1}{\sqrt{\pi at}}\left\{1-\frac{1}{2at}+\frac{1.3}{(2at)^2}-\frac{1.3.5}{(2at)^3}+\cdots\right\}. \tag{91}$$

For large values of t ($at>5$) this series is accurately and rapidly computable. Furthermore it shows by mere inspection the behavior of $V(t)$ for large values of t, and that it ultimately approaches zero as $1/\sqrt{\pi at}$.

Let us now see how Heaviside attacked this problem and how he arrived at a divergent solution from the operational formula. Returning to the operational equation (85), it can be written as

$$V=\frac{\sqrt{p/a}}{1+\sqrt{p/a}} \tag{92}$$

Now expand the denominator by the binomial theorem: we get formally

$$\begin{aligned}V&=\left\{1-\sqrt{\frac{p}{a}}+\frac{p}{a}-\frac{p}{a}\sqrt{\frac{p}{a}}+\left(\frac{p}{a}\right)^2-\cdots\right\}\sqrt{\frac{p}{a}}\\ &=\left(1+\frac{p}{a}+\left(\frac{p}{a}\right)^2+\cdots\right)\sqrt{\frac{p}{a}}\\ &\quad-\left(\frac{p}{a}+\left(\frac{p}{a}\right)^2+\left(\frac{p}{a}\right)^3+\cdots\right).\end{aligned} \tag{93}$$

Heaviside's procedure at this point was as remarkable as it was successful. He first discarded the second series in integral powers of p as meaningless. He then indentified $\sqrt{p}$ with $1/\sqrt{\pi t}$ and replaced p^n by d^n/dt^n in the first series, getting

$$V=\left(1+\frac{1}{a}\frac{d}{dt}+\frac{1}{a^2}\frac{d^2}{dt^2}+\cdots\right)\frac{1}{\sqrt{\pi at}} \tag{94}$$

or, carrying out the indicated differentiation,

$$V=\frac{1}{\sqrt{\pi at}}\left(1-\frac{1}{2at}+\frac{1.3}{(2at)^2}-\frac{1.3.5}{(2at)^3}+\cdot\cdot\right)$$

which agrees with (91).

This is a typical example of a Heaviside divergent solution for which he offered no explanation and no proof other than its practical success. His procedure in this respect is quite unsatisfactory and in particular his discarding an entire series without explanation is intellectually repugnant. We shall leave these questions for the present, however; later we shall make a systematic study of his divergent solutions and rationalize them in a satisfactory manner. First, however, we shall take up a specific problem for which Heaviside obtains a divergent solution without discarding any terms.

Problem C: Current Entering a Line of Distributed L, R and C. Consider a transmission line of distributed inductance L, resistance R, and capacity C per unit length. The differential equations of current and voltage are

$$\begin{aligned}(L\frac{d}{dt}+R)I&=-\frac{\partial}{\partial x}V\\ C\frac{d}{dt}V&=-\frac{\partial}{\partial x}I.\end{aligned} \tag{95}$$

Replacing d/dt by p, we get

$$\begin{aligned}(pL+R)I&=-\frac{\partial}{\partial x}V\\ Cp\cdot V&=-\frac{\partial}{\partial x}I.\end{aligned} \tag{96}$$

Equations (96) correspond exactly with (58) for the non-inductive cable, except that we must replace R by $pL+R$. For the infinitely long line, therefore, the operational formula for the current entering the line is

$$I=\sqrt{\frac{pC}{pL+R}}V_0 \tag{97}$$

where V_0 is the voltage at the line terminals. If this is a "unit e.m.f." we have, as our operational equation,

$$I=\sqrt{\frac{pC}{pL+R}} \tag{98}$$

which can be written as

$$I=\sqrt{\frac{C}{L}}\frac{1}{\sqrt{1+2\lambda/p}} \tag{99}$$

where $\lambda=R/2L$.

The corresponding integral equation is

$$\sqrt{\frac{C}{L}}\frac{1}{\sqrt{p^2+2\lambda p}}=\int_0^\infty e^{-pt}I(t)\,dt. \tag{100}$$

From either equation (99) or (100) and formula (p) of the table of integrals, we see at once that the solution is

$$I=\sqrt{\frac{C}{L}}e^{-\lambda t}I_0(\lambda t) \tag{101}$$

where $I_0(\lambda t)$ is the Bessel function $J_0(i\lambda t)$, where $i=\sqrt{-1}$. (The function is, however, a pure real.)

Heaviside's procedure, in the absence of any correlation between the operational equation and the infinite integral, was quite different. Remarking, with reference to equation (99), that "the suggestion to employ the binomial theorem is obvious," he expands it in the form

$$I=\sqrt{\frac{C}{L}}\left\{1-\frac{\lambda}{p}+\frac{1.3}{2!}\left(\frac{\lambda}{p}\right)^2-\frac{1.3.5}{3!}\left(\frac{\lambda}{p}\right)^3+\cdots\right\} \tag{102}$$

and replaces $1/p^n$ by $t^n/n!$ in accordance with the rule discussed in preceding sections. The explicit solution is then

$$I=\sqrt{\frac{C}{L}}\left\{1-\lambda t+\frac{1.3}{(2!)^2}(\lambda t)^2-\frac{1.3.5}{(3!)^2}(\lambda t)^3+\cdots\right\} \tag{103}$$

a convergent solution in rising powers of t. As yet, however, he does not recognize this series as the power series expansion of (101), which it is. He does, however, recognize the practical impossibility of using it for computing for large values of t, and remarks "But the binomial theorem furnishes another way of expanding the operator (operational equation), viz. in rising powers of p." Thus, returning to (99), it can be written as,

$$I=\sqrt{\frac{C}{L}}\frac{\sqrt{p/2\lambda}}{\sqrt{1+p/2\lambda}}. \tag{104}$$

Now expand the denominator by the binomial theorem: we get

$$I=\sqrt{\frac{C}{L}}\left\{1-\frac{p}{4\lambda}+\frac{1.3}{2!}\left(\frac{p}{4\lambda}\right)^2-\frac{1.3.5}{3!}\left(\frac{p}{4\lambda}\right)^3+\cdots\right\}\sqrt{\frac{p}{2\lambda}}. \quad (105)$$

He now identifies $\sqrt{p/2\lambda}$ with $1/\sqrt{2\pi\lambda t}$ and replaces p^n in the series by d^n/dt^n, thus getting finally

$$I=\sqrt{\frac{C}{L}}\frac{1}{\sqrt{2\pi\lambda t}}\left\{1+\frac{1}{8\lambda t}+\frac{(1.3)^2}{2!(8\lambda t)^2}+\frac{(1.3.5)^2}{3!(8\lambda t)^3}+\ldots\right\}. \quad (106)$$

This series solution is divergent: Heaviside recognizes it, however, as the asymptotic expansion of the function $e^{-\lambda t}I_0(\lambda t)$, and thus arrives at the solution

$$I=\sqrt{\frac{C}{L}}e^{-\lambda t}I_0(\lambda t) \quad (101)$$

which we have obtained from our table of integrals.

Now the divergent expansion (106) is the well known asymptotic expansion of the function $e^{-\lambda t}I_0(\lambda t)$, which is usually derived by difficult and intricate processes. The directness and simplicity with which Heaviside derives it is extraordinary.

We note in this example that no integral powers of p appear in the divergent expansion: consequently no terms are discarded. Otherwise Heaviside's process is as startling and remarkable as in the example discussed in the preceding section.

We shall later encounter many problems in which asymptotic solutions are derivable as in the preceding example. We have sufficient data, however, in these two typical examples to take up a systematic discussion of the theory of Heaviside's divergent solution of the operational equation.[1]

[1] Quite recently an entirely different treatment of this whole problem, based on the Fourier Integral identity, has been developed by Dr. Norbert Wiener. A brief report on this work was made before the American Mathematical Society in May, 1925; it has since been published in *Math. Annalen*, Bd. 95, Heft 4, 1926.

CHAPTER V

THE THEORY OF THE ASYMPTOTIC SOLUTION OF OPERATIONAL EQUATIONS

A study of Heaviside's methods, as exemplified in the preceding examples and in many problems dealt with in his Electromagnetic Theory, Vol II, shows that they may be divided into two classes: (I) those of which the operational equation is of the form

$$h=F(p)\sqrt{p} \tag{I}$$

and (II) those of which the operational equation is of the form

$$h=\phi(p^k\sqrt{p}) \tag{II}$$

where k is an integer.

Heaviside himself does not distinguish between the two classes, but employs the following rule[1] for obtaining asymptotic expansion solutions:

If the operational equation

$$h=1/H(p)$$

can be expanded in the form

$$\begin{aligned} h=a_0+a_1p+a_2p^2+\cdots+a_np^n+\cdots \\ (b_0+b_1p+b_2p^2+\cdots+b_np^n+\cdots)\sqrt{p}, \end{aligned} \tag{107}$$

a solution, usually divergent, is obtained by discarding the first expansion entirely, except for the leading constant terms a_0, replacing $\sqrt{p}$ by $1/\sqrt{\pi t}$ and p^n by d^n/dt^n in the second expansion, whence an explicit series solution results.

$$h=a_0+\left(b_0+b_1\frac{d}{dt}+b_2\frac{d^2}{dt^2}+\cdots\right)\frac{1}{\sqrt{\pi t}} \tag{108}$$

$$=a_0+\frac{1}{\sqrt{\pi t}}\left(b_0-b_1\frac{1}{2t}+b_2\frac{1.3}{(2t)^2}-b_3\frac{1.3.5}{(2t)^3}+\cdots\right). \tag{109}$$

The present chapter is devoted to the mathematical theory of this type of expansion solution of the operational equation, and to an investigation of its practical and theoretical significance.

[1] It does not appear that Heaviside makes a formal statement of this rule. It is a generalization, however, of his procedure in a large number of specific problems.

Heaviside himself gives no information which would serve us as a guide in informing us when the rule is applicable and when it is not. Consequently it becomes a matter of practical importance, not only to investigate the underlying mathematical philosophy of the rule and to establish it on the basis of orthodox mathematics, but also to develop if possible a criterion of its applicability. In this investigation we shall have recourse to the integral equation of the problem.

We shall take up first the type of problem (Class I) in which the operational equation is

$$h=\frac{1}{H(p)}=F(p)\sqrt{p} \tag{110}$$

and assume that $F(p)$ admits of the formal power series expansion

$$F(p)=b_0+b_1p+b_2p^2+b_3p^3+\cdots \tag{111}$$

The corresponding integral equation is

$$\frac{F(p)}{\sqrt{p}}=\int_0^\infty h(t)e^{-pt}\,dt. \tag{112}$$

We now assume the existence of an auxiliary function $k(t)$, defined and determined by the auxiliary integral equation

$$F(p)=\int_0^\infty k(t)e^{-pt}\,dt. \tag{113}$$

Now since

$$\frac{1}{\sqrt{p}}=\int_0^\infty e^{-pt}\frac{dt}{\sqrt{\pi t}} \tag{114}$$

it follows from (112), (113), and (114) and Borel's Theorem, or Theorem IV, that

$$h(t)=\frac{1}{\sqrt{\pi}}\int_0^t\frac{k(\tau)}{\sqrt{t-\tau}}\,d\tau. \tag{115}$$

Now if we differentiate (113) repeatedly with respect to p and put $p=0$, it follows from the expansion (111) that

$$b_n=(-1)^n\int_0^\infty\frac{t^n}{n!}k(t)dt. \tag{116}$$

This equation presupposes, it should be noted, the convergence of the infinite integrals for all values of n, and therefore imposes severe restrictions on $k(t)$ and hence on $F(p)$. We shall suppose that these restrictions are satisfied and discuss them later.

Now (115) can be written as:—

$$h(t)=\frac{1}{\sqrt{\pi t}}\int_0^t d\tau.k(\tau)(1-\tau/t)^{-1/2}. \tag{117}$$

It can be shown that, if $k(t)$ satisfies the restrictions underlying (116), the integral (117) has an asymptotic solution obtained as follows:—Expand the factor $(1-\tau/t)^{-\frac{1}{2}}$ by the binomial theorem, replace the upper limit of integration by ∞, and integrate term by term: thus

$$h(t)\sim\frac{1}{\sqrt{\pi t}}\left\{\int_0^\infty k(t)dt+\frac{1}{2t}\int_0^\infty\frac{t}{1!}k(t)dt\right.$$
$$\left.+\frac{1.3}{(2t)^2}\int_0^\infty\frac{t^2}{2!}k(t)dt+\cdots\right\}. \quad (118)$$

Finally from (116) we get

$$h(t)\sim\frac{1}{\sqrt{\pi t}}\left\{b_0-b_1\frac{1}{2t}+b_2\frac{1.3}{(2t)^2}-b_3\frac{1.3.5}{(2t)^3}+\cdots\right\} \quad (119)$$

which agrees exactly with the Heaviside rule for this case.

The foregoing says nothing regarding the asymptotic character of the solution. It is easy to see qualitatively, however, that (118) and therefore (119) does represent the behavior of the definite integral (117) for large values of t, provided $k(t)$ converges with sufficient rapidity.

The foregoing analysis may now be summarized in the following proposition; which will be called Rule I.

If the operational equation $h=1/H(p)$ is reducible to the form

$$h=F(p)\sqrt{p}$$

and if $F(p)$ admits of power series expansion in p: thus

$$F(p)=b_0+b_1p+b_2p^2+\cdots+b_np^n+\cdots$$

so that formally,

$$h=(b_0+b_1p+b_2p^2+\cdots+b_np^n+\cdots)\sqrt{p}$$

an explicit series solution, usually asymptotic, is obtained by replacing $\sqrt{p}$ by $1/\sqrt{\pi t}$ and p^n(n integral) by d^n/dt^n, whence

$$h(t)\sim\left(b_0+b_1\frac{d}{dt}+b_2\frac{d^2}{dt^2}+\cdots\right)\frac{1}{\sqrt{\pi t}}$$
$$\sim\frac{1}{\sqrt{\pi t}}\left(b_0-b_1\frac{1}{2t}+b_2\frac{1.3}{(2t)^2}-b_3\frac{1.3.5}{(2t)^3}+\cdots\right)$$

provided the function $k=k(t)$, defined by the operational equation $k=F(p)$, and the infinite integrals

$$\int_0^\infty t^n k(t)dt \quad (n=1,2,\ldots.)$$

exist.

We shall now apply the foregoing theory to a physical problem discussed in the last section: namely, the current entering an infinitely long line of inductance L, resistance R and capacity C per unit length. It will be recalled (see equation (100)) that the integral equation of this problem is

$$\sqrt{\frac{C}{L}}\frac{1}{\sqrt{p^2+2\lambda p}}=\int_0^\infty e^{-pt}I(t)dt$$

where $\lambda=R/2L$, and that the solution is

$$I=\sqrt{\frac{C}{L}}e^{-\lambda t}I_0(\lambda t).$$

We can derive the solution in another form appropriate for our purposes by writing

$$\sqrt{\frac{C}{L}}\frac{1}{\sqrt{p}}\frac{1}{\sqrt{p+2\lambda}}=\int_0^\infty e^{-pt}I(t)dt$$

Now since

$$\frac{1}{\sqrt{p}}=\int_0^\infty e^{-pt}\frac{dt}{\sqrt{\pi t}}$$

and

$$\frac{1}{\sqrt{p+2\lambda}}=\int_0^\infty e^{-pt}\frac{e^{-2\lambda t}}{\sqrt{\pi t}}dt$$

it follows from Borel's theorem that

$$I=\sqrt{\frac{C}{L}}\frac{1}{\pi}\int_0^t\frac{e^{-2\lambda\tau}}{\sqrt{\tau}\sqrt{t-\tau}}d\tau.$$

Now subject this definite integral (omitting the factor $\sqrt{C/L}$) to the same process applied to (117): we get

$$\frac{1}{\pi\sqrt{t}}\left\{\int_0^\infty\frac{e^{-2\lambda t}}{\sqrt{t}}dt+\frac{1}{2t}\int_0^\infty\frac{\sqrt{t}}{1!}e^{-2\lambda t}dt\right.$$
$$\left.+\frac{1.3}{(2t)^2}\int_0^\infty\frac{t\sqrt{t}}{2!}e^{-2\lambda t}dt+\cdots\right\}.$$

The infinite integrals are known and have been evaluated. Substituting their values this series becomes:—

$$\frac{1}{\sqrt{2\pi\lambda t}}\left\{1+\frac{1}{8\lambda t}+\frac{1^2.3^2}{2!(8\lambda t)^2}+\frac{1^2.3^2.5^2}{3!(8\lambda t)^3}+\cdots\right\}$$

which is in fact the well known asymptotic expansion of the function $e^{-\lambda t}I_0(\lambda t)$.

A second example may be worth while. Consider the case of an e.m.f. $e^{-\lambda t}$ impressed at time $t=0$ on a cable of distributed

resistance R and capacity C: required the current entering the cable. The required formula is[1]

$$I=\sqrt{\frac{C}{\pi R}}\frac{d}{dt}\int_0^t\frac{e^{-\lambda(t-\tau)}}{\sqrt{\tau}}d\tau$$

$$=\sqrt{\frac{C}{\pi R}}\left\{\frac{1}{\sqrt{t}}-\lambda\int_0^t\frac{e^{-\lambda\tau}}{\sqrt{t-\tau}}d\tau\right\} \qquad (120)$$

by obvious transformations.

Asymptotic expansion of the definite integral as in the preceding example gives the asymptotic formula

$$I=-\sqrt{\frac{C}{\pi Rt}}\left\{\frac{1}{2\lambda t}+\frac{1.3}{(2\lambda t)^2}+\frac{1.3.5}{(2\lambda t)^3}+\cdots\right\}.$$

The operational formula of the problem is

$$I=\sqrt{\frac{C}{R}}\frac{p}{p+\lambda}\sqrt{p}$$

$$=\sqrt{\frac{C}{R}}\frac{p/\lambda}{1+p/\lambda}\sqrt{p}$$

$$=\sqrt{\frac{C}{R}}\left\{\frac{p}{\lambda}-\left(\frac{p}{\lambda}\right)^2+\left(\frac{p}{\lambda}\right)^3-\cdots\right\}\sqrt{p}.$$

Applying the Heaviside Rule, we get the asymptotic expansion

$$I=-\sqrt{\frac{C}{\pi Rt}}\left\{\frac{1}{2\lambda t}+\frac{1.3}{(2\lambda t)^2}+\frac{1.3.5}{(2\lambda t)^3}+\cdots\right\}$$

which agrees with the preceding formula, derived from the definite integral.

We shall now discuss a specific problem in which the Heaviside Rule breaks down. For example let us take the preceding problem, and replace the applied e.m.f. $e^{-\lambda t}$ by $\sin \omega t$. The formula corresponding to (120) is now

$$I=\omega\sqrt{\frac{C}{\pi R}}\int_0^t\frac{\cos\omega\tau}{\sqrt{t-\tau}}d\tau. \qquad (121)$$

If we now attempt to expand the definite integral of (121) in the same way as that of (120), we find that the process breaks down because each component of the infinite integral is now itself infinite. In fact no asymptotic solution of this problem exists.

Let us, however, start with the operational formula: since

$$\int_0^\infty e^{-pt}\sin\omega t dt=\frac{\omega}{p^2+\omega^2}$$

[1] The derivation of the formulas in this problem is left as an exercise for the reader.

it is

$$I=\sqrt{\frac{C}{R}}\frac{\omega p}{p^2+\omega^2}\sqrt{p}.$$

Now expand this in accordance with the Heaviside Rule: we get, operationally,

$$I=\sqrt{\frac{C}{R}}\left\{\left(\frac{p}{\omega}\right)-\left(\frac{p}{\omega}\right)^3+\left(\frac{p}{\omega}\right)^5-\cdots\right\}\sqrt{p}$$

and explicitly

$$I=-\sqrt{\frac{C}{\pi Rt}}\left\{\frac{1}{2\omega t}-\frac{1.3.5}{(2\omega t)^3}+\cdots\right\}$$

which is quite incorrect.[1] The incorrectness of the result will be evident when we remember that the final value of the current is the *steady state* current in response to sin ωt, or

$$\sqrt{\frac{\omega C}{2R}}(\cos\omega t+\sin\omega t). \qquad (122)$$

This result can be derived directly from (121) by writing it as

$$I=\omega\sqrt{\frac{C}{\pi R}}\left\{\cos\omega t\int_0^t\frac{\cos\omega t}{\sqrt{t}}dt+\sin\omega t\int_0^t\frac{\sin\omega t}{\sqrt{t}}dt\right\}.$$

If the time is made indefinitely great the upper limits of the integrals may be replaced by infinity. The infinite integrals are known: substitution of their known values gives (122).

This example illustrates the care which must be used in applying Heaviside's rules for obtaining divergent solutions and the importance of having a method of checking the correctness of his processes and results.

We now take up the discussion of the asymptotic expansion solution of operational equations of the type

$$h=\phi(p^k\sqrt{p}) \qquad (k \text{ integral}). \qquad (123)$$

The Heaviside Rule for this type of operational equation may be formulated as follows (RULE II):

If the operational equation $h=1/H(p)$ is reducible to the form

$$h=\phi(p^k\sqrt{p})$$

and if ϕ admits of power series expansion in the argument, thus

$$h=a_0+a_1p^k\sqrt{p}+a_2p^{2k+1}+a_3p^{3k+1}\sqrt{p}+\cdots$$

[1] While this series is incorrect as an asymptotic expansion of the current it has important significance, as we shall see, in connection with the building up of alternating currents.

a series solution, usually divergent and asymptotic, is obtained by discarding integral powers of p, and writing

$$h \sim a_0 + (a_1 p^k + a_3 p^{3k+1} + a_5 p^{3k+2} + \ldots)\sqrt{p}$$

The explicit series solution then results from replacing $\sqrt{p}$ *by* $1/\sqrt{\pi t}$*, and* p^n *by* d^n/dt^n*, whence*

$$h \sim a_0 + \left(a_1 \frac{d^k}{dt^k} + a_3 \frac{d^{3k+1}}{dt^{3k+1}} + a_5 \frac{d^{3k+2}}{dt^{3k+2}} + \ldots\right)\frac{1}{\sqrt{\pi t}}$$

$$\sim a_0 + \frac{(-1)^k}{\sqrt{\pi t}}\left(a_1 \frac{1.3\ldots(2k-1)}{(2t)^k} - a_3 \frac{1.3\ldots(6k+1)}{(2t)^{3k+1}} + \ldots\right).$$

The theory of this series solution will be based on the following proposition, deducible from the identity $\int_0^\infty \frac{e^{-pt}}{\sqrt{\pi t}}dt = 1/\sqrt{p}$:

If the function $F(p)$ *of the integral equation*

$$F(p) = \int_0^\infty f(t)e^{-pt}dt$$

approaches $1/\sqrt{p}$ *as* p *approaches zero, then* $f(t)$ *ultimately behaves as* $1/\sqrt{\pi t}$*; that is, if* $F(p) \to 1/\sqrt{p}$ *as* $p \to 0$*, then* $f(t) \sim 1/\sqrt{\pi t}$ *as* $t \to \infty$*, provided that* $f(t)$ *converges to zero, and contains no term or factor which is ultimately oscillatory.*

To illustrate what this condition means suppose that

$$f(t) = \frac{a}{\sqrt{\pi t}} + \frac{b \cos \omega t}{\sqrt{\pi t}}$$

then

$$\int_0^\infty f(t)e^{-pt}dt \to a/\sqrt{p} \text{ as } p \to 0,$$

and the oscillatory term in $f(t)$ converges to a higher order. The presence of such oscillatory terms vitiate, therefore, the Heaviside Rule: in the following discussion we shall assume that they are absent.

We are now prepared to discuss the operational equation

$$h = \phi(p^k\sqrt{p})$$

and for convenience shall assume that $k = 0$ so that the operational equation becomes

$$h = \phi(\sqrt{p})$$

of which the corresponding or equivalent integral equation is

$$\frac{1}{p}\phi(\sqrt{p}) = \int_0^\infty h(t)e^{-pt}dt. \qquad (123b)$$

We assume that $\phi(\sqrt{p})$ admits of formal power series expansion in the argument: thus

$$\phi(\sqrt{p})=a_0+a_1\sqrt{p}+a_2p+a_3p\sqrt{p}+a_4p^2+\cdots$$

without, however, implying anything regarding the convergence of this expansion.

We now introduce the series of auxiliary functions, $g, g_1, g_2, g_3, \ldots$ defined by the following scheme

$$\begin{aligned}
g(t)&=h(t)-a_0\\
g_1(t)&=g(t)-\frac{a_1}{\sqrt{\pi t}}\\
g_2(t)&=t.g_1(t)+\frac{1}{2}\frac{a_3}{\sqrt{\pi t}}\\
g_3(t)&=t.g_2(t)-\frac{1.3}{2^2}\frac{a_5}{\sqrt{\pi t}}\\
g_4(t)&=t.g_3(t)+\frac{1.3.5}{2^3}\frac{a_7}{\sqrt{\pi t}}\\
&\cdots\cdots
\end{aligned}\tag{123c}$$

Successive substitutions in the integral equation (123b) and repeated differentiations with respect to p, lead to the set of formulas,

$$\begin{aligned}
&\int_0^\infty g(t)e^{-pt}\,dt\sim\frac{a_1}{\sqrt{p}}\text{ as }p\to 0\\
&\int_0^\infty t.g_1(t)e^{-pt}\,dt\sim-\frac{a_3}{2\sqrt{p}}\text{ as }p\to 0\\
&\int_0^\infty t.g_2(t)e^{-pt}\,dt\sim\frac{1.3}{2^2}\frac{a_5}{\sqrt{p}}\text{ as }p\to 0\\
&\int_0^\infty t.g_3(t)e^{-pt}\,dt\sim-\frac{1.3.5}{2^3}\frac{a_7}{\sqrt{p}}\text{ as }p\to 0\\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\tag{123d}$$

Now assuming that $h(t)$ satisfies the restrictions stated in the preceding proposition, it follows from that proposition, that

$$\begin{aligned}
g(t)&\sim a_1/\sqrt{\pi t}\text{ as }t\to\infty\\
g_1(t)&\sim-\frac{a_3}{2t\sqrt{\pi t}}\text{ as }t\to\infty\\
g_2(t)&\sim\frac{1.3}{2^2t}\frac{a_5}{\sqrt{\pi t}}\text{ as }t\to\infty\\
g_3(t)&\sim-\frac{1.3.5}{2^3t}\frac{a_7}{\sqrt{\pi t}}\text{ as }t\to\infty
\end{aligned}\tag{123e}$$

From the set of equations (123d) and (123e) it follows by successive substitutions that

$$h(t) \sim a_0 + \frac{1}{\sqrt{\pi t}}\left(a_1 - a_3\frac{1}{2t} + a_5\frac{1.3}{(2t)^2} - a_7\frac{1.3.5}{(2t)^3} + \cdots\right)$$

which agrees with the series gotten by applying the Heaviside Rule.

The defect of this derivation, which, however, appears to be inherent, is that it requires us to know or assume at the outset that $h(t)$ satisfies the required restrictions. Consequently an automatic application of the Heaviside Rule may or may not give correct results. On the other hand if we know that an expansion solution in inverse fractional powers of t exists, the Heaviside Rule gives the series with extraordinary directness and simplicity.

The type of expansion solution just discussed will now be illustrated by some specific problems. The first problem is that of the propagated voltage in the non-inductive cable in response to a "unit e.m.f." It will be recalled that in a preceding chapter we derived the operational formula

$$V = e^{-\sqrt{\alpha p}} \tag{124}$$

where $\alpha = x^2RC$, for the voltage at distance x from the terminal of a non-inductive cable of distributed resistance R and capacity C, in response to a "unit e.m.f." impressed at point $x = 0$. Heaviside's solution of this operational equation proceeds as follows:

Expansion of the exponential function in the usual power series gives

$$V = 1 - \frac{\sqrt{\alpha p}}{1!} + \frac{\alpha p}{2!} - \frac{\alpha p\sqrt{\alpha p}}{3!} + \frac{(\alpha p)^2}{4!} - \cdots$$

which may be rearranged as

$$V = 1 - \left(1 + \frac{\alpha p}{3!} + \frac{(\alpha p)^2}{5!} + \cdots\right)\sqrt{\alpha p} + \left(\frac{\alpha p}{2!} + \frac{(\alpha p)^2}{4!} + \frac{(\alpha p)^3}{6!} + \cdots\right) \tag{125}$$

Heaviside now discards the series in integral powers of p entirely, replaces $\sqrt{p}$ by $1/\sqrt{\pi t}$ and p^n by d^n/dt^n in the first series, and thus gets

$$V = 1 - \left(1 + \frac{\alpha}{3!}\frac{d}{dt} + \frac{\alpha^2}{5!}\frac{d^2}{dt^2} + \cdots\right)\sqrt{\frac{\alpha}{\pi t}}$$

$$= 1 - \sqrt{\frac{\alpha}{\pi t}}\left\{1 - \frac{1}{3!}\left(\frac{\alpha}{2t}\right) + \frac{1.3}{5!}\left(\frac{\alpha}{2t}\right)^2 - \frac{1.3.5}{7!}\left(\frac{\alpha}{2t}\right)^3 + \cdots\right\} \tag{126}$$

or

$$V=1-\sqrt{\frac{\alpha}{\pi t}}\left(1-\frac{1}{3}\left(\frac{\alpha}{4t}\right)+\frac{1}{5.2!}\left(\frac{\alpha}{4t}\right)^2-\frac{1}{7.3!}\left(\frac{\alpha}{4t}\right)^3+\cdots\right). \tag{127}$$

This solution is correct, as will be shown subsequently.

A rather remarkable feature of this solution—a point on which Heaviside makes no comment—is that it is absolutely convergent. In other words, a process of expansion which in other problems leads to a divergent or asymptotic solution, here results in a convergent series expansion.

To verify this solution we start with the corresponding integral equation of the problem

$$\frac{1}{p}e^{-\sqrt{\alpha p}}=\int_0^\infty V(t)e^{-pt}\,dt. \tag{128}$$

It follows from this formula and theorem (II) that

$$V(t)=\int_0^t \phi(t)\,dt$$

where $\phi(t)$ is determined by the integral equation

$$e^{-\sqrt{\alpha p}}=\int_0^\infty \phi(t)e^{-pt}\,dt.$$

Now from formula (f) of the table of integrals

$$e^{-\sqrt{\alpha p}}=\frac{1}{2}\sqrt{\frac{\alpha}{\pi}}\int_0^\infty e^{-pt}\frac{e^{-\alpha/4t}}{t\sqrt{t}}\,dt$$

whence

$$\phi(t)=\frac{1}{2}\sqrt{\frac{\alpha}{\pi}}\,\frac{e^{-\alpha/4t}}{t\sqrt{t}}$$

and finally

$$V(t)=\frac{1}{\sqrt{\pi}}\int_0^{t'}\frac{e^{-1/\tau}}{\tau\sqrt{\tau}}\,d\tau, \text{ where } t'=4t/\alpha. \tag{129}$$

To convert this to the form of (127) we write

$$V(t)=\frac{1}{\sqrt{\pi}}\int_0^\infty\frac{e^{-1/\tau}}{\tau\sqrt{\tau}}\,d\tau-\frac{1}{\sqrt{\pi}}\int_{t'}^\infty\frac{e^{-1/\tau}}{\tau\sqrt{\tau}}\,d\tau. \tag{130}$$

The value of the infinite integral is known to be $\sqrt{\pi}$ so that

$$V=1-\frac{1}{\sqrt{\pi}}\int_{t'}^\infty\frac{e^{-1/\tau}}{\tau\sqrt{\tau}}\,d\tau. \tag{131}$$

Now in the integral term of (131) expand $e^{-1/\tau}$ in the usual exponential power series and then integrate term by term; the series solution (127) results. This series, while absolutely con-

vergent, is difficult to compute for small values of t; an asymptotic expansion, which can be employed for computation for small values of t is gotten as follows:

Write (129) as

$$V=\frac{1}{\sqrt{\pi}}\int_0^{t'}\sqrt{\tau}\,de^{-1/\tau}$$
$$=\sqrt{\frac{t'}{\pi}}e^{-1/t'}-\frac{1}{2\sqrt{\pi}}\int_0^{t'}\frac{e^{-1/\tau}}{\sqrt{\tau}}d\tau.$$

Repeated partial integrations of this type lead to the series

$$V=\sqrt{\frac{t'}{\pi}}e^{-1/t'}\left\{1-\left(\frac{t'}{2}\right)+1.3\left(\frac{t'}{2}\right)^2-\cdots\right\}. \tag{132}$$

It is interesting to note, in passing, that an asymptotic solution of this type does not appear to be directly deducible from the operational equation. We observe also that, in this problem, the series in inverse powers of t is convergent while the series in ascending powers of t is divergent: the converse is the case in the problems discussed previously.

A second specific problem may be stated as follows:

Let a "unit e.m.f." be impressed on an infinitely long non-inductive cable of distributed resistance R and capacity C per unit length through a terminal resistance R_0: required the voltage V on the cable terminals. The formulation of the operational equation of this problem is very simple. It will be recalled that the operational formula for the current entering the cable with terminal voltage V is $V\sqrt{Cp/R}$. But the current is clearly also equal to $(1-V)/R_0$: equating these expressions we get

$$\frac{1-V}{R_0}=V\sqrt{pC/R}$$

whence

$$V=\frac{1}{\sqrt{p/\lambda}+1} \tag{133}$$

where $1/\sqrt{\lambda}=R_0\sqrt{C/R}$. This is the required operational formula.

To derive the Heaviside divergent expansion, expand (133) by the binomial theorem: thus

$$\begin{aligned}V&=1-\sqrt{p/\lambda}+(p/\lambda)-(p/\lambda)^{3/2}+\cdots\\&=1-(1+p/\lambda+(p/\lambda)^2+\cdots)\sqrt{p/\lambda}\\&\quad+(p/\lambda+(p/\lambda)^2+(p/\lambda)^3+\cdots).\end{aligned} \tag{134}$$

Discard the second series in integral powers of p; replace $\sqrt{p}$ by $1/\sqrt{\pi t}$ and p^n by d^n/dt^n in the first series, thus getting

$$V=1-\left(1+\frac{1}{\lambda}\frac{d}{dt}+\frac{1}{\lambda^2}\frac{d^2}{dt^2}+\cdots\right)\frac{1}{\sqrt{\pi\lambda t}} \tag{135}$$

$$=1-\frac{1}{\sqrt{\pi\lambda t}}\left(1-\frac{1}{2\lambda t}+\frac{1.3}{(2\lambda t)^2}-\cdots\right) \tag{136}$$

which is the asymptotic solution of the problem.

To verify this solution we shall consider the more general operational equation

$$h=\frac{1}{p^n\sqrt{p}+1} \qquad (n \text{ integral}) \tag{137}$$

a form of equation to which a number of fairly important problems is reducible. (The parameter λ of equation (133) can be eliminated from explicit consideration by means of theorem VI.)

Multiplying numerator and denominator of equation (137) by $p^n\sqrt{p}-1$, it becomes

$$h=\frac{p^n\sqrt{p}-1}{p^{2n+1}-1}=\frac{p^n}{p^{2n+1}-1}\sqrt{p}-\frac{1}{p^{2n+1}-1} \tag{138}$$

and by direct partial fraction expansion, this is equivalent to

$$h=\frac{\sqrt{p}}{2n+1}\sum_{m=0}^{2n}\frac{p_m^{n+1}}{p-p_m}-\frac{1}{2n+1}\sum_{m=0}^{2n}\frac{p_m}{p-p_m} \tag{139}$$

where

$$p_m=e^{i\frac{2m\pi}{2n+1}} \quad (m=0,1,2\ldots 2n).$$

Write, for convenience,

$$h=\sum_{m=0}^{2n}h_m$$

and consider the operational equation

$$h_m=\frac{1}{2n+1}\left(\frac{p_m^{n+1}}{p-p_m}\sqrt{p}-\frac{p_m}{p-p_m}\right). \tag{140}$$

By the rules of the operational calculus, fully discussed in preceding chapters, the solution of this is

$$h_m(t)=\frac{1}{2n+1}\left(\frac{p_m^{n+1}}{\sqrt{\pi}}\int_0^t\frac{e^{p_m(t-\tau)}}{\sqrt{\tau}}d\tau+1-e^{p_m t}\right). \tag{141}$$

We have now to distinguish two cases: (1) when the *real part* of p_m is positive, and (2) when the real part is negative.

Taking up case (1) first, the preceding can be written

$$h_m(t)=\frac{1}{2n+1}\left(1+e^{p_m t}\left\{\frac{p_m^{n+1}}{\sqrt{\pi}}\int_0^t \frac{e^{p_m\tau}}{\sqrt{\tau}}d\tau-1\right\}\right) \tag{142}$$

$$=\frac{1}{2n+1}\left(\begin{array}{l}1+e^{p_m t}\left\{\dfrac{p_m^{n+1}}{\sqrt{\pi}}\displaystyle\int_0^\infty \frac{e^{-p_m\tau}}{\sqrt{\tau}}d\tau-1\right\}\\ -\dfrac{p_m^{n+1}}{\sqrt{\pi}}e^{p_m t}\displaystyle\int_t^\infty \frac{e^{-p_m\tau}}{\sqrt{\tau}}d\tau\end{array}\right) \tag{143}$$

$$=\frac{1}{2n+1}\left(1-\frac{p_m^{n+1}}{\sqrt{\pi}}e^{p_m t}\int_t^\infty \frac{e^{-p_m\tau}}{\sqrt{\tau}}d\tau\right). \tag{144}$$

Repeated integration by parts of the definite integral leads to an asymptotic series, identical with that obtained by applying the Heaviside Rule to the operational equation (140).

If, on the other hand, the *real part* of p_m is negative, we write (141) as

$$h_m(t)=\frac{1}{2n+1}\left(\begin{array}{l}1-e^{p_m t}\\ +\dfrac{p_m^{n+1}}{\sqrt{\pi}}\displaystyle\int_0^t \frac{e^{p_m\tau}}{\sqrt{t-\tau}}d\tau\end{array}\right). \tag{145}$$

The term $e^{p_m t}$ ultimately dies away, and the definite integral can be expanded asymptotically in accordance with the theory discussed under Rule 1, again leading to an asymptotic series identical with that given by direct application of the Heaviside Rule to the operational equation.

Consequently since the operational equation in h_m can be asymptotically expanded by means of the Heaviside Rule, the operational equation in $h=\sum h_m$ is similarly asymptotically expandible, and the Heaviside Rule is verified for equation (137).

We have now covered, more or less completely, the theoretical rules and principles of the operational calculus in so far as they can be formulated in general terms. We shall now apply these principles and rules to the solution of important technical problems relating to the propagation of current and voltage along lines. In doing so, while we shall take advantage of our table of integrals with the corresponding solutions of the operational equation, we shall also sketch Heaviside's own methods of solution.

We shall close this discussion of divergent and asymptotic expansions with a general expansion solution of considerable

theoretical and practical importance in the problem of the building-up of alternating currents. It will be recalled from Theorem VIII that the response of a network of generalized operational impedance $H(p)$ to an e.m.f. $E(t)$ impressed at time $t=0$ is given by the operational formula

$$x=\frac{V(p)}{H(p)}$$

where $E=V(p)$ is the operational equation of the applied e.m.f.: that is, analytically

$$\frac{1}{p}V(p)=\int_0^\infty E(t)e^{-pt}\,dt.$$

Now suppose that the impressed e.m.f. is $\sin \omega t$: then by formula (h) of the table of integrals

$$V(p)=\frac{\omega p}{p^2+\omega^2} \tag{146}$$

and denoting x by x_s

$$x_s=\frac{\omega p}{p^2+\omega^2}\frac{1}{H(p)}. \tag{147}$$

If, on the other hand, the impressed e.m.f. is $\cos \omega t$, then by formula (i)

$$V(p)=\frac{p^2}{p^2+\omega^2} \tag{148}$$

and

$$x=x_c=\frac{p^2}{p^2+\omega^2}\frac{1}{H(p)}. \tag{149}$$

Now let us consider the operational expansion suggested by the Heaviside processes:

$$\begin{aligned} x_s&=\frac{p}{\omega}\left(1+\frac{p^2}{\omega^2}\right)^{-1}\frac{1}{H(p)}\\ &=\left\{\frac{p}{\omega}-\left(\frac{p}{\omega}\right)^3+\left(\frac{p}{\omega}\right)^5-\left(\frac{p}{\omega}\right)^7+\cdots\right\}\frac{1}{H(p)} \end{aligned} \tag{150}$$

and

$$\begin{aligned} x_c&=\left(\frac{p}{\omega}\right)^2\left(1+\frac{p^2}{\omega^2}\right)^{-1}\frac{1}{H(p)}\\ &=\left\{\left(\frac{p}{\omega}\right)^2-\left(\frac{p}{\omega}\right)^4+\left(\frac{p}{\omega}\right)^6-\left(\frac{p}{\omega}\right)^8+\cdots\right\}\frac{1}{H(p)}. \end{aligned} \tag{151}$$

Now let us identify $1/H(p)$ with $h(t)$ and replace p^n by d^n/dt^n: we get

$$x_s=\left\{\frac{1}{\omega}\frac{d}{dt}-\frac{1}{\omega^3}\frac{d^3}{dt^3}+\frac{1}{\omega^5}\frac{d^5}{dt^5}-\cdots\right\}h(t) \tag{152}$$

and

$$x_c=\left\{\frac{1}{\omega^2}\frac{d^2}{dt^2}-\frac{1}{\omega^4}\frac{d^4}{dt^4}+\frac{1}{\omega^6}\frac{d^6}{dt^6}-\cdots\right\}h(t). \tag{153}$$

We have now to inquire into the significance of equations (152) and (153), derived from the operational equations for the response of the system to an e.m.f. $\sin \omega t$ and $\cos \omega t$ respectively, impressed at time $t=0$. From the mode of derivation of these expansions from the operational equations it might be inferred that they are the divergent or asymptotic expansions of the operational equations (147) and (149). This would certainly not be an unreasonable inference in the light of the Heaviside expansions we have just been considering. This inference is, however, not correct: on the other hand, the series (152) and (153) have a definite physical significance, as we shall now show from the explicit equations of the problem.

By equation (31), the explicit equation for x_s, given operationally by (147), is

$$x_s=\frac{d}{dt}\int_0^t \sin \omega\tau . h(t-\tau)d\tau=\int_0^t \sin \omega(t-\tau)h'(\tau)d\tau+h(0)\sin \omega t \tag{154}$$

where $h'(t)=d/dt\, h(t)$. By a well known trigonometric formula, this is

$$x_s=\sin \omega t\int_0^t \cos \omega t . h'(t)dt-\cos \omega t\int_0^t \sin \omega t . h'(t)dt+h(0)\sin \omega t.$$

Writing

$$\int_0^t dt=\int_0^\infty dt-\int_t^\infty dt$$

this becomes

$$x_s=\sin \omega t\int_0^\infty \cos \omega t . h'(t)dt-\cos \omega t\int_0^\infty \sin \omega t . h'(t)dt$$
$$+h(0)\sin \omega t-\int_t^\infty \sin \omega(t-\tau)h'(\tau)d\tau. \tag{155}$$

The first three terms are simply the steady state response to the impressed e.m.f. $\sin \omega t$: that is, they represent the ultimate steady state value of x_s when the transient oscillations have

died away. The last term, which we shall denote by T_s, represents the transient oscillations which are set up when the e.m.f. is applied. Thus

$$T_s = -\int_t^\infty \sin \omega(t-\tau)h'(\tau)d\tau. \tag{156}$$

Now from (156)

$$T_s = -\frac{1}{\omega}\int_t^\infty h'(\tau).d.\cos \omega(\tau-t)$$

and integrating by parts

$$T_s = \frac{1}{\omega}\frac{d}{dt}h(t) + \frac{1}{\omega}\int_t^\infty \cos \omega(\tau-t)\frac{d^2}{d\tau^2}h(\tau)d\tau. \tag{157}$$

Repeating the process of partial integration, we get:

$$T_s = \frac{1}{\omega}\frac{d}{dt}h(t) - \frac{1}{\omega^2}\int_t^\infty \sin \omega(\tau-t)\frac{d^3}{d\tau^3}h(\tau)d\tau. \tag{158}$$

Repeating the process again

$$T_s = \frac{1}{\omega}\frac{d}{dt}h(t) - \frac{1}{\omega^3}\frac{d^3}{dt^3}h(t) + \frac{1}{\omega^4}\int_t^\infty \sin \omega(\tau-t)\frac{d^5}{d\tau^5}h(\tau)d\tau.$$

This process can be repeated indefinitely, and we get

$$T_s = \left(\frac{1}{\omega}\frac{d}{dt} - \frac{1}{\omega^3}\frac{d^3}{dt^3} + \frac{1}{\omega^5}\frac{d^5}{dt^5} - \cdots + \frac{(-1)^{n-1}}{\omega^{2n-1}}\frac{d^{2n-1}}{dt^{2n-1}}\right)h(t)$$
$$+ \frac{(-1)^n}{\omega^{2n}}\int_t^\infty \sin \omega(\tau-t)\frac{d^{2n+1}}{dt^{2n+1}}h(\tau)d\tau. \tag{159}$$

The series expansion (159), except for the remainder term, is identical with the series expansion (152) derived directly from the operational equation. This series may be either convergent or divergent, depending on the frequency $\omega/2\pi$ and the character of the indicial admittance function $h(t)$. In the important problems of the building-up of alternating currents in cables and lines we shall see that, even when divergent, the series is of an asymptotic character and can be employed for computation.

We thus arrive at the following theorem:

If an e.m.f. $\sin \omega t$ is impressed at time $t=0$ on a network or system of generalized indicial admittance $h(t)$, and if the *transient distortion*, T_s, is defined as the instantaneous difference between the actual response of the system and the steady state response, then T_s can be expressed as the series,

$$\left(\frac{1}{\omega}\frac{d}{dt} - \frac{1}{\omega^3}\frac{d^3}{dt^3} + \frac{1}{\omega^5}\frac{d^5}{dt^5} - \cdots + \frac{(-1)^{n-1}}{\omega^{2n-1}}\frac{d^{2n-1}}{dt^{2n-1}}\right)h(t) \tag{160}$$

with a remainder term

$$\frac{(-1)^n}{\omega^{2n}}\int_t^\infty \sin\omega(\tau-t)\frac{d^{2n+1}}{dt^{2n+1}}h(\tau)d\tau.$$

If the impressed e.m.f. is cos ωt, the corresponding series for the transient distortion, T_c, is

$$\left(\frac{1}{\omega^2}\frac{d^2}{dt^2}-\frac{1}{\omega^4}\frac{d^4}{dt^4}+\frac{1}{\omega^6}\frac{d^6}{dt^6}-\cdots-\frac{(-1)^n}{\omega^{2n}}\frac{d^{2n}}{dt^{2n}}\right)h(t) \qquad (161)$$

with a remainder term

$$\frac{(-1)^n}{\omega^{2n}}\int_t^\infty \cos\omega(\tau-t)\frac{d^{2n+1}}{dt^{2n+1}}h(\tau)d\tau.$$

The second part of this theorem, relating to the transient distortion, T_c, in response to an e.m.f. cos ωt, is derived from formula (31) by processes precisely analagous to those employed above in deriving the series expansion for T_s. The derivation will be left to the reader.

To summarize the preceding discussion of the divergent solution of operational equations, it may be said that the theory is as yet rather unsatisfactory. To the physicist it is unsatisfactory because he requires an automatic rule giving a correct asymptotic expansion by purely algebraic operations without investigations of remainder terms or auxiliary functions. Furthermore, the precise sense in which the expansion asymptotically represents the solution cannot be stated in general, but requires an independent investigation in the case of each individual problem.

On the other hand, when an asymptotic expansion is known to exist, the Heaviside Rule finds this expansion with incomparable directness and simplicity, the problem of justifying the expansion being a purely mathematical one, which usually need not trouble the physicist. Furthermore, on the purely mathematical side, the Heaviside Rule is of large interest and should lead to interesting developments in the theory of asymptotic expansions.

APPENDIX TO CHAPTER V

It will have been observed that the asymptotic expansions derivable from the operational equation by the direct application of the Heaviside Rules, as discussed in the preceding chapter, are all of the form:

$$a_0+\frac{1}{\sqrt{\pi t}}(b_0+b_1/t+b_2/t^2+b_3/t^3+\cdots)$$

In other words, they are all simple series in inverse fractional powers of t. In addition to this limitation, the direct application of the Heaviside Rules, as we have seen, does not always lead to correct results. In this appendix we shall briefly discuss some extensions and modifications of the direct Heaviside method whereby the limitations, inherent in the direct application of the Heaviside Rules, may often be successfully avoided.

The asymptotic solution of operational equations is of great importance from the standpoint of both pure and applied mathematics. Its importance in practical applications to technical and physical problems does not require emphasis, while from the mathematical standpoint it not only is of value in the solution of infinite integrals of the type $\int_0^\infty f(t)e^{-pt}\,dt$, but promises to be a useful tool in the general theory of asymptotic expansions.

I. We start with the operational equation

$$h = \frac{F(p)}{G(p)} \tag{1}$$

and assume that $F(p)$ admits of the formal series expansion

$$F(p) = b_0 + b_1 p + b_2 p^2 + b_3 p^3 + \cdots \tag{2}$$

It is evident that equation (1) is a generalization of (110) of the preceding chapter, into which it degenerates when $1/G(p)$ is put equal to $\sqrt{p}$. A consideration of (1) will therefore lead to a formal generalization of Rule I.

Let the function $g = g(t)$ be defined by the operational equation

$$g = 1/G(p) \tag{3}$$

and an auxiliary function $k = k(p)$ by the operational equation

$$k = p \cdot F(p). \tag{4}$$

Assuming that these functions exist, we have by Borel's theorem

$$h(t) = \int_0^t k(\tau)g(t-\tau)\,d\tau \tag{5}$$

Now returning to (1) let us write, by virtue of the expansion (2),

$$h = (a_0 + a_1 p + a_2 p^2 + \cdots)\frac{1}{G(p)} \tag{6}$$

Now, guided by the usual Heaviside procedure, let us replace $1/G(p)$ by $g(t)$, in accordance with (3), and replace p^n by d^n/dt^n in the series factor. The right hand side of (6) then becomes

$$\left(a_0+a_1\frac{d}{dt}+a_2\frac{d^2}{dt^2}+\cdots\right)g(t) \tag{7}$$

We have now to inquire into the significance of this expansion.

Expand the function $g(t-\tau)$ as a power series in τ, and substitute in (5): we get

$$\begin{aligned} h(t)=g(t)\int_0^t k(\tau)d\tau-g'(t)\int_0^t\frac{\tau}{1!}k(\tau)d\tau \\ +\cdots+(-1)^n g^{(n)}(t)\int_0^t\frac{\tau^n}{n!}k(\tau)d\tau \\ +\int_0^t k(\tau)R_n(t-\tau)d\tau \end{aligned} \tag{8}$$

where $R_n(t-\tau)$ is the remainder in the expansion of $g(t-\tau)$.

We now suppose that $k(t)$ converges in such a way that the integrals

$$\int_0^\infty\frac{\tau^n}{n!}k(\tau)d\tau \qquad (n=1,2,3\,.\,.n)$$

exist. On this assumption it follows from the integral equation

$$\int_0^\infty k(t)e^{-pt}dt=F(p)$$

and the series expansion (2), that

$$\int_0^\infty\frac{\tau^n}{n!}k(\tau)d\tau=(-1)^n b_n \tag{9}$$

Now writing

$$\int_0^t=\int_0^\infty-\int_t^\infty$$

equation (8) becomes by virtue of (9),

$$\begin{aligned} h(t)=\left(b_0+b_1\frac{d}{dt}+b_2\frac{d^2}{dt^2}+\cdots+b_n\frac{d^n}{dt^n}\right)g(t) \\ +\int_0^t k(\tau)R_n(t-\tau)d\tau \\ -\int_t^\infty d\tau\cdot k(\tau)\left\{1-\frac{\tau}{1!}\frac{d}{dt}+\cdots+(-1)^n\frac{\tau^n}{n!}\frac{d^n}{dt^n}\right\}g(t) \end{aligned} \tag{10}$$

The first series of (10) is identical with (7), while the rest of the right hand side of (10) represent remainder terms. The question as to whether the series (7) represents the function $h(t)$, asymptotically or otherwise, can only be answered by an independent investigation of the remainder terms in each specific problem. The expansion (10) therefore formulates a generalization of the problem underlying Rule I, into which it degenerates when $g(t)=1/\sqrt{\pi t}$. The fact that no general statement can be made as to the way in which the series (7) represents $h(t)$ indicates the care which must be taken in the interpretation of the operators. Often, however, the expansion (7) is valid and quite useful: in addition it possesses considerable theoretical interest.

II. If the variable $x(t)$ is defined by the definite integral

$$x(t)=\frac{d}{dt}\int_0^t e^{\lambda(t-\tau)}h(\tau)d\tau \tag{2.1}$$

it is known that, provided the real part of λ is not negative, it may be written as

$$x(t)=\frac{e^{\lambda t}}{H(\lambda)}+y(t) \tag{2.2}$$

Furthermore, in a number of important cases the component $y(t)$ admits of asymptotic expansion. On the other hand, the operational equation

$$x=\frac{p}{p-\lambda}\frac{1}{H(p)} \tag{2.3}$$

corresponding to (2.1) does not admit of direct asymptotic expansion and the direct application of the Heaviside Rule is invalid. We can, however, set up an operational equation for $y(t)$ which leads to the required asymptotic expansion, where such an expansion exists. The procedure is as follows:

From (2.2) and (2.3) we have the usual integral equation:

$$\int_0^\infty dt\, e^{-pt}\left[y(t)+\frac{1}{H(\lambda)}e^{\lambda t}\right]=\frac{1}{p-\lambda}\frac{1}{H(p)}$$

Integrating the second term in the integral and rearranging gives

$$\int_0^\infty dt\, e^{-pt}y(t)=\frac{1}{p-\lambda}\left[\frac{1}{H(p)}-\frac{1}{H(\lambda)}\right]$$

and the corresponding operational equation is

$$y=\frac{p}{p-\lambda}\left[\frac{1}{H(p)}-\frac{1}{H(\lambda)}\right] \tag{2.4}$$

As an example, let $\lambda = i\omega$ and $h(t) = 1/\sqrt{\pi t}$. The problem is then that discussed on page 66 and formulated by equation (121), concerning which it was remarked that the straightforward procedure gave incorrect results. In this case

$$1/H(p) = \sqrt{p}; \qquad 1/H(\lambda) = \sqrt{i\omega}$$

and the operational equation is

$$y = \frac{p}{p - i\omega}[\sqrt{p} - \sqrt{i\omega}]$$
$$= \frac{p}{\sqrt{p} + \sqrt{i\omega}} = \frac{p}{\sqrt{i\omega}}\left(1 + \frac{\sqrt{p}}{\sqrt{i\omega}}\right)^{-1}$$

or, expanding in powers of $\sqrt{p}$,

$$y = \frac{1}{\sqrt{i\omega}}\left(p - \frac{p\sqrt{p}}{\sqrt{i\omega}} + \frac{p^2}{(i\omega)} - \frac{p^2\sqrt{p}}{(i\omega)\sqrt{i\omega}} + \cdots\right)$$

Dropping the integral powers of p, according to Rule II (p. 67) and replacing $\sqrt{p}$ by $1/\sqrt{\pi t}$ and p^n by d^n/dt^n gives

$$y \sim i\left\{\frac{1}{\omega}\frac{d}{dt} - \frac{1}{\omega^3}\frac{d^3}{dt^3} + \cdots\right\}\frac{1}{\sqrt{\pi t}} + \left\{\frac{1}{\omega^2}\frac{d^2}{dt^2} - \frac{1}{\omega^4}\frac{d^4}{dt^4} + \cdots\right\}\frac{1}{\sqrt{\pi t}} \tag{2.5}$$

which is correct.

Note that, if in the last equation we generalize $1/\sqrt{\pi t}$ into $h(t)$, the series becomes the equivalent of (160) and (161) for the transient distortion in the case of sinusoidal impressed forces.

III. In a number of important problems, the function h, defined by the operational equation $h = 1/H(p)$, has an asymptotic expansion of the form

$$h \sim \frac{e^{\lambda t}}{\sqrt{\pi t}}(a_0 + a_1/t + a_2/t^2 + \cdots) \tag{3.1}$$

In such cases the direct application of the Heaviside rules breaks down, due to the presence of the exponential factor. If, however, the *form* of the asymptotic expansion is known, the following procedure enables us to derive the asymptotic expansion by operational methods.

Let us write

$$h(t) = e^{\lambda t} g(t) \tag{3.2}$$

and substitute in the integral equation: we get

$$\int_0^\infty dt\, e^{-pt} e^{\lambda t} g(t) = \frac{1}{pH(p)}$$

and, by Theorem Va,

$$\int_0^\infty dt\, e^{-pt} g(t) = \frac{1}{(p+\lambda)H(p+\lambda)} \tag{3.3}$$

whence, operationally,

$$g = \frac{p}{(p+\lambda)H(p+\lambda)}. \tag{3.4}$$

If the assumed form of expansion exists, then g must admit of expansion in inverse fractional powers of t, and this expansion must be derivable from the operational equation by the direct application of the Heaviside rules.

Example

As an interesting example of the preceding we shall derive the usual asymptotic expansion of the Bessel function $J_0(\lambda t)$ from the operational equation (n) and the corresponding integral equation

$$\int_0^\infty dt\, e^{-pt} J_0(\lambda t) = \frac{1}{\sqrt{p^2+\lambda^2}}. \tag{3.5}$$

It is known that the function $J_0(\lambda t)$ can be represented as

$$u(t) \cos \lambda t + v(t) \sin \lambda t \tag{3.6}$$

and that u and v admit of asymptotic expansions in inverse fractional powers of t.

Write (3.6) as

$$\frac{1}{2} e^{i\lambda t}[u(t) - iv(t)] + \frac{1}{2} e^{-i\lambda t}[u(t) + iv(t)] \tag{3.7}$$

and substitute in (3.5): we get

$$\begin{aligned} &\frac{1}{2}\int_0^\infty dt\, e^{-pt} e^{i\lambda t}[u(t) - iv(t)] \\ &+\frac{1}{2}\int_0^\infty dt\, e^{-pt} e^{-i\lambda t}[u(t) + iv(t)] = \frac{1}{\sqrt{p^2+\lambda^2}}, \end{aligned} \tag{3.8}$$

whence, replacing $p - i\lambda$ by q and then replacing q by p, we get:

$$\begin{aligned} &\frac{1}{2}\int_0^\infty dt\, e^{-pt}[u(t) - iv(t)] \\ &+\frac{1}{2}\int_0^\infty dt\, e^{-pt} e^{-i2\lambda t}[u(t) + iv(t)] = \frac{1}{\sqrt{p^2+i2\lambda p}} \end{aligned} \tag{3.9}$$

Now the function

$$e^{-i2\lambda t}[u(t) + iv(t)]$$

is oscillatory and does not admit of asymptotic expansion in inverse fractional powers of t. Consequently if the right hand

side of (3.9) is expanded in fractional powers of p, we shall get therefrom by the procedure discussed in Chapter V the asymptotic expansion of the function

$$\frac{1}{2}[u(t)-iv(t)]$$

That is to say, *in an asymptotic sense only*, we can set up the operational equation

$$\begin{aligned} u-iv &= \frac{2p}{\sqrt{p^2+i2\lambda p}} \qquad (3.10)\\ &= \frac{2}{\sqrt{i2\lambda}}\left[1-i\frac{p}{2\lambda}\right]^{-\frac{1}{2}}\sqrt{p} \\ &= e^{-i\frac{\pi}{4}}\sqrt{\frac{2}{\lambda}}\left[1-i\frac{p}{2\lambda}\right]^{-\frac{1}{2}}\sqrt{p} \end{aligned}$$

The factor $e^{-i\pi/4}$ suggests that $u-iv$ be replaced by $e^{-i\pi/4}(P-iQ)$, whence

$$P-iQ=\sqrt{\frac{2}{\lambda}}\left[1-i\frac{p}{2\lambda}\right]^{-\frac{1}{2}}\sqrt{p} \qquad (3.11)$$

Expanding the right hand side by the Binomial Theorem, replacing p^n by d^n/dt^n and $\sqrt{p}$ by $1/\sqrt{\pi t}$, we get:

$$P-iQ\sim\sqrt{\frac{2}{\pi\lambda t}}\left[1-i\frac{1}{8\lambda t}-\frac{1^2.3^2}{2!(8\lambda t)^2}\right.$$
$$\left.+i\frac{1^2.3^2.5^2}{3!(8\lambda t)^3}+\frac{1^2.3^2.5^2.7^2}{4!(8\lambda t)^4}-\cdots\right] \qquad (3.12)$$

whence

$$\begin{aligned} P &\sim\sqrt{\frac{2}{\pi\lambda t}}\left[1-\frac{1^2.3^2}{2!(8\lambda t)^2}+\cdots\cdot\right] \\ Q &\sim\sqrt{\frac{2}{\pi\lambda t}}\left[\frac{1}{8\lambda t}-\frac{1^2.3^2.5^2}{3!(8\lambda t)^3}+\cdots\cdot\right] \end{aligned} \qquad (3.13)$$

and finally

$$J_0(\lambda t)\sim P\cos\left(\lambda t-\frac{\pi}{4}\right)+Q\sin\left(\lambda t-\frac{\pi}{4}\right) \qquad (3.14)$$

which is the usual asymptotic expansion.

The foregoing discussion is tentative and far from complete. It does, however, suggest that important developments of the theory of asymptotic expansion may result from a study of the operational equation or rather its integral equation equivalent.

CHAPTER VI

PROPAGATION OF CURRENT AND VOLTAGE IN THE NON-INDUCTIVE CABLE

The principal practical applications of the operational calculus in electrotechnics are to the theory of the propagation of current and voltage along transmission systems. Of such transmission systems the simplest is the non-inductive cable. The theory of the non-inductive cable is not only of great historic interest, relating as it does to Kelvin's early work on the possibility of transatlantic telegraphy, but is also of very considerable practical importance today, and serves as a basis for the theory of submarine telegraphy over long distances. We shall therefore consider the propagation phenomena in the non-inductive cable in some detail.

The propagation phenomena in any type of transmission system are isolated and exhibited in the clearest possible manner when we confine attention to the infinitely long line, with voltage applied directly to the line terminals. Furthermore, as we shall see later, the solution for the infinitely long line is fundamental and can be extended to the more practical case of the finite line with terminal impedances. We therefore, in this chapter, shall confine our attention to the case of the infinitely long cable with voltage applied directly to the cable terminals.

Consider a cable of distributed resistance R and capacity C per unit length, extending from $x=0$ along the positive x axis. From a previous chapter (see equations (64) and (65), we are in possession of the operational equations of voltage and current; they are, for the infinitely long line,

$$V=e^{-\sqrt{\alpha p}}V_0, \tag{162}$$

$$I=\frac{1}{Rx}\sqrt{\alpha p}\, e^{-\sqrt{\alpha p}}\, V_0=\sqrt{\frac{Cp}{R}}e^{-\sqrt{\alpha p}}\, V_0, \tag{163}$$

where $\alpha=x^2RC$, and V_0 is the terminal cable voltage at $x=0$. Let us now assume that the terminal voltage V_0 is a "unit e.m.f.;" then

$$V=e^{-\sqrt{\alpha p}}, \tag{164}$$

$$I=\frac{1}{Rx}\sqrt{\alpha p}\, e^{-\sqrt{\alpha p}}. \tag{165}$$

The solution of (164) for V was considered in some detail in the preceding chapter; it is, by (129)

$$V=\frac{1}{\sqrt{\pi}}\int_0^\tau \frac{e^{-1/\tau}}{\tau\sqrt{\tau}}d\tau \tag{166}$$

where $\tau=4t/\alpha=4t/x^2RC$. Series expansions of this solution were also given. Another equivalent form is, by (131)

$$V=1-\frac{2}{\sqrt{\pi}}\int_0^{1/\sqrt{\tau}} e^{-\tau^2}d\tau. \tag{167}$$

This last form, recognizable also from inspection of the series expansion (132), is useful because the integral term is what is called the error function and has been completely computed and tabulated.

Before discussing these formulas and the light they throw on propagation phenomena in the non-inductive cable, we shall derive the solution for the current. A very simple way of doing this is to make use of the differential equation (57)

$$I=-\frac{1}{R}\frac{\partial}{\partial x}V.$$

Now from (166) and the relation

$$\frac{\partial}{\partial x}=\frac{d\tau}{dx}\frac{d}{d\tau}$$

we get

$$\begin{aligned}\frac{\partial}{\partial x}V&=\frac{1}{\sqrt{\pi}}\frac{e^{-1/\tau}}{\tau\sqrt{\tau}}\frac{d}{dx}\frac{4t}{x^2RC}\\&=-\frac{2}{x\sqrt{\pi}}\frac{e^{-1/\tau}}{\sqrt{\tau}},\end{aligned}$$

whence

$$I=\frac{2}{xR\sqrt{\pi}}\frac{e^{-1/\tau}}{\sqrt{\tau}}=\sqrt{\frac{C}{\pi Rt}}e^{-1/\tau}. \tag{168}$$

It is worthwhile verifying the formula by direct solution from the operational equation (165). From formula (g) of the table of integrals, we have

$$\begin{aligned}h&=e^{-2\sqrt{\lambda p}}\sqrt{p}\sqrt{\frac{C}{R}}\\&=\frac{e^{-\lambda/t}}{\sqrt{\pi t}}\sqrt{\frac{C}{R}}.\end{aligned}$$

Comparison with the operational equation shows that they are identical, within a constant factor provided we put $\lambda = \alpha/4$. Consequently the solution of (165) is

$$I = \sqrt{\frac{C}{\pi R t}}\, e^{-\alpha/4t} = \sqrt{\frac{C}{\pi R t}}\, e^{-1/\tau}$$

which agrees with (168). This, it may be remarked, is an excellent example of the utility of the table of integrals in solving operational equations.

This formula is easily calculated for large values of t by expanding the exponential function; it is

$$\frac{2}{Rx}\frac{1}{\sqrt{\pi\tau}}\left[1 - \left(\frac{1}{\tau}\right) + \frac{1}{2!}\left(\frac{1}{\tau}\right)^2 - \cdots\right].$$

The propagation phenomena of the non-inductive cable are therefore determined by the pair of equations

$$V = \frac{1}{\sqrt{\pi}}\int_0^{\tau} \frac{e^{-1/\tau}}{\tau\sqrt{\tau}}\, d\tau = 1 - \frac{2}{\sqrt{\pi}}\int_0^{1/\sqrt{\tau}} e^{-\tau^2} d\tau \qquad (169)$$

and

$$I = \frac{2}{\sqrt{\pi}\, xR}\,\frac{e^{-1/\tau}}{\sqrt{\tau}} = \sqrt{\frac{C}{\pi R t}}\, e^{-1/\tau} \qquad (170)$$

where $\tau = 4t/\alpha = \dfrac{4t}{x^2 RC}$.

Now an important feature of these formulas is that the voltage at point x is a function only of $\dfrac{4}{x^2 RC}\,t$; that is, of $4t$ *divided by the total resistance and capacity* of the cable from $x = 0$ to $x = x$. The same statement holds for the *form* of the current wave: its magnitude, however, is inversely proportional to xR, or the total resistance of the cable up to point x. Consequently a single curve, with proper time scale serves to give the voltage wave at any point on the cable. Similarly a single curve, with proper time and amplitude scales, serves to depict the current wave at any distance from the cable terminals. These curves are given in Figs. 3 and 4.

Referring to the curve depicting the current wave, we observe that it is finite for all values of $t > 0$; consequently, in the ideal cable, the velocity of propagation is infinite. This is a consequence, of course, of the fact that the distributed inductance of the cable is neglected. Actually, of course, the velocity of

propagation cannot exceed the velocity of light. The error, however, in neglecting the inductance in the case of long cables is appreciable only near the head of the wave provided we confine attention to d.c. or low frequency voltages. This point will be discussed and explained more fully in connection with the transmission line.

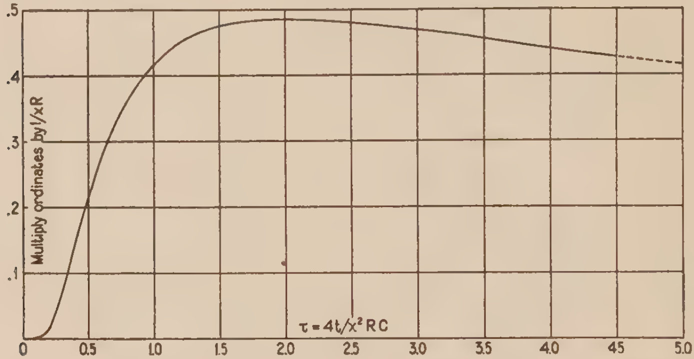

FIG. 3.—Current in non-inductive cable ($G=0$) unit e.m.f. applied.

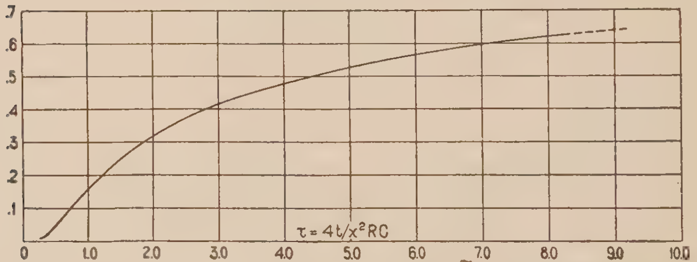

FIG. 4.—Voltage in non-inductive cable ($G=0$) unit e.m.f. applied.

The current, while finite, is negligibly small until τ reaches the value 0.2. In the neighborhood of this point it begins to build up rapidly; reaches at $\tau=2$ its maximum value

$$\frac{2}{\sqrt{\pi}\,xR}\frac{e^{-0.5}}{\sqrt{2}}=\frac{2}{\sqrt{\pi}\,xR}(0.429)$$

and then begins to decrease, ultimately dying away in accordance with the formula

$$\frac{2}{\sqrt{\pi}\,xR}\frac{1}{\sqrt{\tau}}\left\{1-\frac{1}{\tau}+\frac{1}{2!}\left(\frac{1}{\tau}\right)^2-\cdots\right\}.$$

Its subsidence to its final zero value is very slow; for example, when $\tau=100$ its value is still

$$\frac{2}{\sqrt{\pi}\,xR}(0.10).$$

Turning to the voltage curve, Fig. 4, we see that it is negligibly small until τ reaches the value 0.25, at which point it begins to build up. Its maximum rate of building up occurs when $\tau=2/3$,

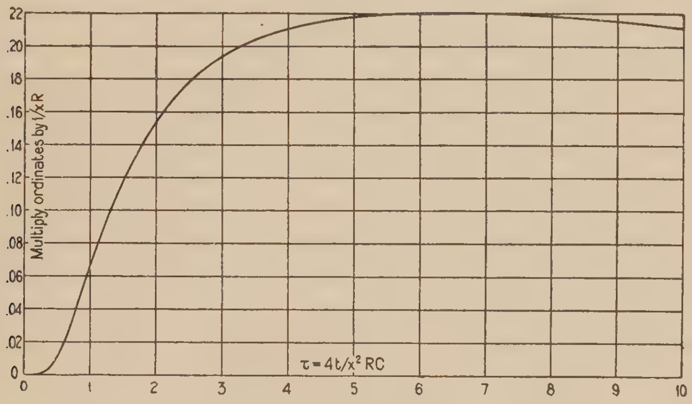

FIG. 5.—Power transmitted in non-inductive cable ($G=0$).

after which it builds up more and more slowly. Its approach to its final steady value is in accordance with the formula

$$V=1-\frac{2}{\sqrt{\pi\tau}}\left(1-\frac{1}{3\tau}+\frac{1}{2!}\frac{1}{5\tau^2}-\cdots\right).$$

Even, therefore, when τ is as great as 100, V differs sensibly from its ultimate value, unity, its value being 0.8876.

Since the actual time is $\frac{x^2RC}{4}\tau$, it follows that the speed of building up is inversely proportional to the square of the length of the cable.

The power curve V·I is given in Fig. 5. V·I is the rate at which energy is being transmitted past the point x of the cable.

The fact that the form of the current and voltage waves depends only on $4t/x^2RC$ is at the basis of Kelvin's famous "KR" law, long applied to cable telegraphy and sometimes incorrectly applied to telephony. When the first transatlantic telegraph cable was under consideration, Kelvin attacked the problem of propagation along the non-inductive cable and arrived at for-

mulas equivalent to (169) and (170). From these formulas he announced the law that the "speed" of the cable, i.e., the number of signals transmissible per unit time, is inversely proportional to the product of the total capacity and total resistance of the cable (KR in the English notation). To see just what this means requires a little digression into the elementary theory of telegraph transmission.

Telegraph signals are transmitted in code by means of "dots" and "dashes." The "dot" is the signal which results when a battery is impressed on the cable for a definite interval of time, after which the cable is short circuited. A "dash" is the same except that the time interval during which the battery is connected to the cable is increased. The "dots" and "dashes" are separated by intervals, called "spaces," during which the cable is short circuited. Now when the cable is short-circuited we may imagine a negative battery impressed on the cable in series with the original battery. Consequently the current in the cable, corresponding to a signal composed of a series of dots, dashes and spaces, will be represented by a series of the form

$$I(t)-I(t-t_1)+I(t-t_2)-I(t-t_3)+I(t-t_4)-\cdot\cdot\cdot \qquad (171)$$

where, in the cable under consideration, $I(t)$ is given by (168). t_1 is the duration of the first impulse, t_2-t_1 of the first space, t_3-t_2 of the second impulse, etc.

Now by (168)

$$I(t)=\frac{2}{xR\sqrt{\pi}}\frac{e^{-1/\tau}}{\sqrt{\tau}}=\frac{2}{xR\sqrt{\pi}}\phi(\tau).$$

τ is, of course, $4t/x^2CR=4t/KR$ (in the English notation). Now suppose that

$$\tau_1=\frac{4t_1}{x^2CR},$$

$$\tau_2=\frac{4t_2}{x^2CR}, \text{ etc.}$$

Then the signal can be written as

$$\frac{2}{xR\sqrt{\pi}}\{\phi(\tau)-\phi(\tau-\tau_1)+\phi(\tau-\tau_2)-\cdot\cdot\cdot\}. \qquad (172)$$

Now if the relative time intervals τ_1, τ_2 . . . are kept constant (as the length of the cable is varied), the actual time intervals t_1, t_2 . . . are proportional to x^2CR or to KR, and the wave form of the total signal is independent of KR, when referred to the relative time scale τ. Hence, if T is the total time of the

signal, T is proportional to x^2CR (or to KR). That is to say, if the duration of the component dots, dashes, spaces of the signal are proportional to the "KR" of the cable, the wave form of the received signal, referred to the τ time scale, is invariable, and the total time required to transmit the signal is proportional to the "KR" of the cable. Now the maximum theoretical speed of transmission on the cable is limited by the requirement that the received signal shall bear a recognizable likeness to the original system of dots and dashes: in other words there is a maximum allowable departure in wave form between received and transmitted signals. If, therefore, the actual speeds of two cables

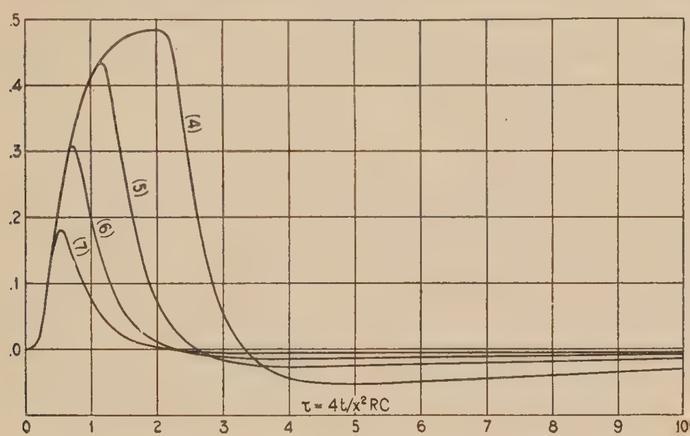

FIG. 6.—Elementary telegraph signals in non-inductive cable.

are inversely proportional to their "KRs," the wave form will be the same. This establishes Kelvin's "KR" law. As a corollary, if the length of the cable is doubled the speed of signaling is reduced to one-quarter, assuming the same definition of signals.

The foregoing will be somewhat clearer, perhaps, if we refer to curves, 4, 5, 6, 7 of Fig. 6 which illustrate the distortion suffered by elementary dot signals in cable transmission. Curve 4 shows the dot signal produced by a unit battery applied to the cable terminals for a time interval $t=2\frac{x^2RC}{4}$, while curves 5, 6 and 7 are the corresponding dot signals when the battery is applied for the time intervals $\frac{x^2RC}{4}$, $\frac{1}{2}\frac{x^2RC}{4}$ and $\frac{1}{4}\frac{x^2RC}{4}$. Any further decrease in the duration of the impressed dot, beyond that

shown in curve 7, does not affect the *shape* of the transmitted dot, which means that the cable speed has reached its theoretical maximum. These curves, it should be observed, can be interpreted in two ways. First, we can regard the length x of the cable as fixed and the duration of the impressed dot as varied. On the other hand, we can regard the actual duration of the impressed dot as constant and the length of the cable as varied. From the latter standpoint the curves illustrate the progressive distortion of the signal as it is transmitted along the cable.

The dot signal of relative duration T can be written as

$$\begin{aligned} D &= I(\tau), \qquad \tau < T \\ &= I(\tau) - I(\tau - T), \qquad \tau > T \end{aligned}$$

and the second expression can be expanded in a Taylor's series, giving

$$D = T\frac{d}{d\tau}I(\tau) - \frac{T^2}{2!}\frac{d^2}{d\tau^2}I(\tau) + \cdots$$

If T is sufficiently short this becomes

$$D = T \cdot I'(\tau). \tag{173}$$

Hence when the dot signal is of sufficiently short relative duration T, the wave shape of the received signal is constant, $I'(\tau)$, and its amplitude is proportional to the relative duration of the dot.

This can be generalized for any type of transmission system: Let the dot signal be produced by an e.m.f. $f(t)$ of actual duration T. Then the received dot signal, by formula (31), is

$$\begin{aligned} D &= \frac{d}{dt}\int_0^t f(\tau)I(t-\tau)d\tau, \qquad t < T \\ &= \frac{d}{dt}\int_0^T f(\tau)I(t-\tau)d\tau, \qquad t > T. \end{aligned}$$

For $t > T$ this becomes

$$D = I'(t)\int_0^T f(\tau)d\tau - I''(t)\int_0^T \tau f(\tau)d\tau + \cdots$$

and for sufficiently short duration T, we have approximately,

$$D = I'(t)\int_0^T f(\tau)d\tau. \tag{174}$$

Hence for a sufficiently short duration of the impressed e.m.f. the received dot signal is of constant wave form, independent of the shape of the impressed e.m.f., and its amplitude is proportional to the time integral of the impressed e.m.f. These principles are of considerable practical importance in telegraphy.

The *leaky cable*, that is, a cable with distributed leakage conductance G in addition to resistance R and capacity C, is of some interest. The differential equations of the problem are given in equations (70); the operational formulas for the case of voltage directly impressed on the terminals of the infinitely long line are

$$V=e^{-x\sqrt{CRp+RG}}\,V_0,$$

$$I=\sqrt{\frac{pC}{R}+\frac{G}{R}}\,e^{-x\sqrt{CRp+RG}}\,V_0.$$

Writing $CRx^2=\alpha$ and $RGx^2=\beta$, $G/C=\lambda$, and assuming a "unit e.m.f." impressed on the cable, this becomes

$$V=e^{-\sqrt{\alpha p+\beta}},\tag{175}$$

$$I=\sqrt{\frac{C}{R}}\sqrt{p+\lambda}\,e^{-\sqrt{\alpha p+\beta}}.\tag{176}$$

These equations are readily solved by means of the table and formulas given in a preceding chapter.

But first let us attempt to solve the operational equation (175) for the voltage by Heaviside methods, guided by the solution of the operational equation

$$V=e^{-\sqrt{\alpha p}}\tag{124}$$

of the preceding chapter. Expand the exponential function in (175) in the usual power series; it is

$$V=1-\sqrt{\alpha p+\beta}+\frac{(\alpha p+\beta)}{2!}-\frac{(\alpha p+\beta)\sqrt{\alpha p+\beta}}{3!}+\cdots\tag{177}$$

Now discard the integral terms and write

$$V=1-\left\{1+\frac{\alpha p+\beta}{3!}+\frac{(\alpha p+\beta)^2}{5!}+\cdots\right\}\sqrt{\alpha p+\beta}.\tag{178}$$

We have now to interpret the expression $\sqrt{\alpha p+\beta}$. We have by ordinary algebra

$$\begin{aligned}\sqrt{\alpha p+\beta}&=\left(1+\frac{\beta}{\alpha p}\right)^{1/2}\sqrt{\alpha p}=\left(1+\frac{\lambda}{p}\right)^{1/2}\sqrt{\alpha p}\\&=\left[1+\frac{\lambda}{2p}-\frac{1}{2!}\left(\frac{\lambda}{2p}\right)^2+\frac{1.3}{3!}\left(\frac{\lambda}{2p}\right)^3+\cdots\right]\sqrt{\alpha p}.\end{aligned}\tag{179}$$

Now identify $\sqrt{p}$ with $1/\sqrt{\pi t}$ in accordance with the Heaviside rule, and $1/p$ with $\int dt$. We get

$$\sqrt{\alpha p+\beta}=\sqrt{\frac{\alpha}{\pi t}}\left\{1+\frac{\lambda t}{1!}-\frac{(\lambda t)^2}{3!}+\frac{1.4}{5!}(\lambda t)^3-\cdots\right\}.\tag{180}$$

Now in the terms of the expansion (178) identify p^n with d^n/dt^n and substitute (180); we get

$$V=1-\left\{1+\frac{1}{3!}\left(\alpha\frac{d}{dt}+\beta\right)+\frac{1}{5!}\left(\alpha^2\frac{d^2}{dt^2}+2\alpha\beta\frac{d}{dt}+\beta^2\right)+\cdots\right\}$$
$$\times\sqrt{\frac{\alpha}{\pi t}}\left\{1+\frac{\lambda t}{1!}-\frac{(\lambda t)^2}{3!}+1.4\frac{(\lambda t)^3}{5!}-\cdots\right\} \tag{181}$$

This series is hopelessly complicated to either interpret or compute. It is, in fact, an excellent illustration of the grave disadvantages under which many of Heaviside's series solutions labor. We shall therefore attack the solution by aid of the theorems and formulas of Chapter IV. The simplicity of the solution which results is remarkable.

The operational formula for the voltage is

$$V=e^{-\sqrt{\alpha p+\beta}}. \tag{175}$$

Now the operational formula for the voltage in the non-leaky cable is (see equation (164))

$$V=e^{-\sqrt{\alpha p}}.$$

In order to distinguish between the two cases, let us denote the voltage in the latter case by V^0; thus

$$V^0=e^{-\sqrt{\alpha p}}. \tag{182}$$

Now by theorem (Va) and equation (182) we have

$$\begin{aligned} V^0e^{-\lambda t}&=\frac{p}{p+\lambda}e^{-\sqrt{\alpha(p+\lambda)}},\\ &=\frac{p}{p+\lambda}e^{-\sqrt{\alpha p+\beta}}. \end{aligned} \tag{183}$$

Now write (175) as

$$\begin{aligned} V&=\frac{p+\lambda}{p}\cdot\frac{p}{p+\lambda}e^{-\sqrt{\alpha p+\beta}},\\ &=\left(1+\frac{\lambda}{p}\right)\cdot\frac{p}{p+\lambda}e^{-\sqrt{\alpha p+\beta}}. \end{aligned} \tag{184}$$

It follows at once by comparison with (183) and the rule that $1/p$ is to be replaced by $\int dt$, that

$$V=\left(1+\lambda\int_0^t dt\right)V^0e^{-\lambda t}. \tag{185}$$

By a precisely similar procedure with the operational formula (176) for the current, we get

$$I=\left(1+\lambda\int_0^t dt\right)I^0e^{-\lambda t} \tag{186}$$

where I^0 is the current in the non-leaky cable. Now by formulas (169) and (170)

$$V^0 = \frac{1}{\sqrt{\pi}} \int_0^{4t/\alpha} \frac{e^{-1/t}}{t\sqrt{t}} dt, \tag{169}$$

$$I^0 = \sqrt{\frac{C}{\pi R t}} e^{-\alpha/4t}, \tag{170}$$

which completes the formal solution of the problem.

Formulas (185) and (186) are extremely interesting, first as showing the superiority of the definite integral to the series expansion—compare (185) with the series expansions (181)—and secondly as exhibiting clearly the effect of leakage on the propagated waves of current and voltage. We see that in both the current and voltage the effect of leakage is two-fold; first it attenuates the wave by the factor $e^{-\lambda t}$, $(\lambda = G/C)$, and secondly it adds a component consisting of the progressive integral of the attenuated wave. This, it may be remarked, is the general effect of leakage in all types of transmission systems. Its effect is, therefore, easily computed and interpreted.

Formulas (185) and (186) are very easy to compute with the aid of a planimeter or integraph; or, failing these devices, by numerical integration. However, for large values of t, the character of the waves is more clearly exhibited if we make use of the identity

$$\int_0^t dt = \int_0^\infty dt - \int_t^\infty dt$$

whence

$$V = \left(1 + \lambda \int_0^\infty dt\right) V^0 e^{-\lambda t} - \lambda \int_t^\infty V^0 e^{-\lambda t} dt \tag{187}$$

and

$$I = \left(1 + \lambda \int_0^\infty dt\right) I^0 e^{-\lambda t} - \lambda \int_t^\infty I^0 e^{-\lambda t} dt. \tag{188}$$

The first two terms of these formulas are clearly the ultimate steady state values of the voltage and current waves, and can be determined by evaluating the infinite integrals. A far simpler and more direct way, however, is to make use of the fact that the ultimate steady values of V and I are gotten from the operational formulas by setting $p = 0$. That this statement is true is easily seen if we reflect that the steady d.c. voltage and current are gotten from the original differential equations of the problem by assuming a steady state and setting $d/dt = 0$.

From the operational formulas we get, therefore

$$\left(1+\lambda\int_0^\infty dt\right)V^0 e^{-\lambda t}=e^{-\sqrt{\beta}}=e^{-x\sqrt{RG}}, \tag{189}$$

$$\left(1+\lambda\int_0^\infty dt\right)I^0 e^{-\lambda t}=\sqrt{\frac{C\lambda}{R}}e^{-\sqrt{\beta}}=\sqrt{\frac{G}{R}}e^{-\sqrt{RG}}. \tag{190}$$

Introducing these expressions into (187) and (188) respectively, we get

$$V=e^{-x\sqrt{RG}}-\lambda\int_t^\infty V^0 e^{-\lambda t}dt, \tag{191}$$

$$I=\sqrt{\frac{G}{R}}e^{-x\sqrt{RG}}-\lambda\int_t^\infty I^0 e^{-\lambda t}dt. \tag{192}$$

The definite integrals can be expanded by partial integration; thus

$$-\lambda\int_t^\infty V^0 e^{-\lambda t}dt=\int_t^\infty V^0 de^{-\lambda t}$$

$$=-V^0 e^{-\lambda t}-\int_t^\infty e^{-\lambda t}\frac{d}{dt}V^0 dt.$$

Continuing this process we get

$$V=e^{-x\sqrt{RG}}-e^{-\lambda t}\left(1+\frac{d}{\lambda dt}+\frac{d^2}{\lambda^2 dt^2}+\cdot\cdot\right)V^0, \tag{193}$$

$$I=\sqrt{\frac{G}{R}}e^{-x\sqrt{RG}}-e^{-\lambda t}\left(1+\frac{d}{\lambda dt}+\frac{d^2}{\lambda^2 dt^2}+\cdot\cdot\right)I^0. \tag{194}$$

Using the values of V^o and I^o, as given by (169) and (170), it is extremely easy to compute V and I, for large values of t, from (193) and (194).

So far we have considered the current and voltage waves in response to a "unit e.m.f.," impressed on the cable at $x=0$. It is of interest and importance to examine the waves due to sinusoidal e.m.fs., suddenly impressed on the cable, particularly in view of proposals to employ alternating currents in cable telegraphy.

We start with the fundamental formula

$$x(t)=\frac{d}{dt}\int_0^t f(t-\tau)h(\tau)d\tau$$

$$=\int_0^t f(t-\tau)h'(\tau)d\tau,$$

provided $h(0)=0$, which is the case in the cable.

If $f(t) = \sin \omega t$, we write

$$
\begin{aligned}
x_s(t) = \sin \omega t \int_0^t \cos \omega t . h'(t) dt \\
- \cos \omega t \int_0^t \sin \omega t . h'(t) dt.
\end{aligned}
\tag{194-a}
$$

Similarly, if the impressed e.m.f. is $\cos \omega t$,

$$
\begin{aligned}
x_c(t) = \cos \omega t \int_0^t \cos \omega t . h'(t) dt \\
+ \sin \omega t \int_0^t \sin \omega t . h'(t) dt.
\end{aligned}
\tag{194-b}
$$

The investigation of the building-up of alternating currents and voltages, therefore, depends on the progressive integrals

$$
\begin{aligned}
C = \int_0^t \cos \omega t . h'(t) dt, \\
S = \int_0^t \sin \omega t . h'(t) dt.
\end{aligned}
\tag{194-c}
$$

For the case of the *voltage* wave on the non-inductive, non-leaky cable these integrals, by aid of equations (169), become, if we write $\omega' = \alpha\omega/4$,

$$
\begin{aligned}
C = \frac{1}{\sqrt{\pi}} \int_0^\tau \frac{e^{-1/\tau} \cos \omega' \tau}{\tau \sqrt{\tau}} d\tau, \\
S = \frac{1}{\sqrt{\pi}} \int_0^\tau \frac{e^{-1/\tau} \sin \omega' \tau}{\tau \sqrt{\tau}} d\tau,
\end{aligned}
\tag{194-d}
$$

where, as before, $\tau = 4t/\alpha$.

For the *current* wave we have, by (170),

$$
\begin{aligned}
C = \frac{2}{\sqrt{\pi}\, xR} \int_0^\tau \left(\frac{1}{\tau} - \frac{1}{2} \right) \frac{e^{-1/\tau} \cos \omega' \tau}{\tau \sqrt{\tau}} d\tau. \\
S = \frac{2}{\sqrt{\pi}\, xR} \int_0^\tau \left(\frac{1}{\tau} - \frac{1}{2} \right) \frac{e^{-1/\tau} \sin \omega' \tau}{\tau \sqrt{\tau}} d\tau.
\end{aligned}
\tag{194-e}
$$

For small values of τ and ω' these integrals can be numerically evaluated without great labor. Mechanical devices, such as the Coradi Harmonic Analyzer, are here of great assistance. In fact the Coradi Analyzer gives these progressive integrals automatically. It may be said, therefore, that a complete mathematical investigation of the building-up of alternating current and voltage waves in the non-inductive cable presents no serious difficulties, although the labor of computation is necessarily

considerable. One fact makes the complete investigation much less laborious than might be supposed. This is, if the foregoing integrals are calculated for a given value of ω', the results apply to all lengths of cable and all actual frequencies $\omega/2\pi$, such that $\alpha\omega$ is a constant. Thus if we double the length of the cable and quarter the frequency, the integrals are unaffected.

The solid curve of Fig. 7 shows the building-up of the cable voltage in response to an e.m.f. $\cos \omega t$, impressed at time $t=0$. The frequency $\omega/2\pi$ is so chosen that $\omega' = \alpha\omega/4 = 2\pi$, and the curve is calculated from equations (194-b) and (194-e). The dotted

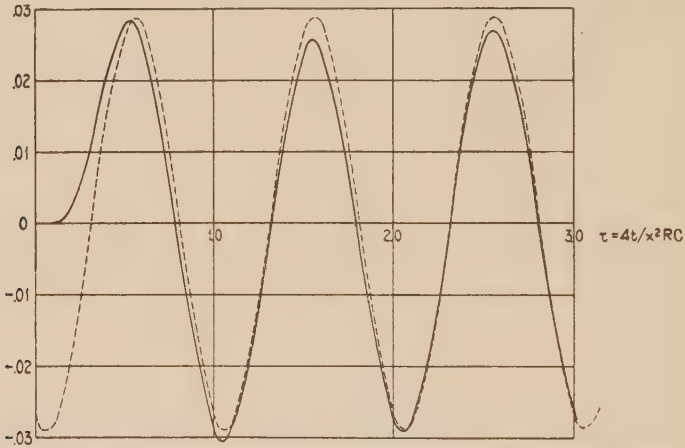

FIG. 7.—Non-inductive cable ($G=0$), building-up of alternating current. Applied e.m.f. $\cos \omega t$; $\omega = 2\pi \dfrac{4}{x^2RC}$.

curve shows the corresponding *steady state* voltage on the cable; that is, the voltage which would exist if the e.m.f. $\cos \omega t$ had been applied at a long time preceding $t=0$. We observe that, for this frequency, the building-up is effectually accomplished in about one cycle, and that the transient distortion is only appreciable during the first half-cycle.

The case is very much different when a higher frequency is applied. Fig. 8 shows the building-up of the alternating current in the cable when an e.m.f. $\sin \omega t$ is applied at time $t=0$. The frequency is so chosen that $\omega' = \alpha\omega/4 = 10\pi$. The outstanding features of this curve are that the initial current surge is very large compared with the final steady state, and that the transient

distortion is relatively very large. It is evident that the frequency here shown could not be employed for signaling purposes. This curve has been computed from the steady state formulas, and equations (160) and (161) for the transient distortion.

If the applied frequency $\omega/2\pi$ is very high, the steady state becomes negligibly small, and the complete current is obtained to a good approximation by taking the leading terms of (160)

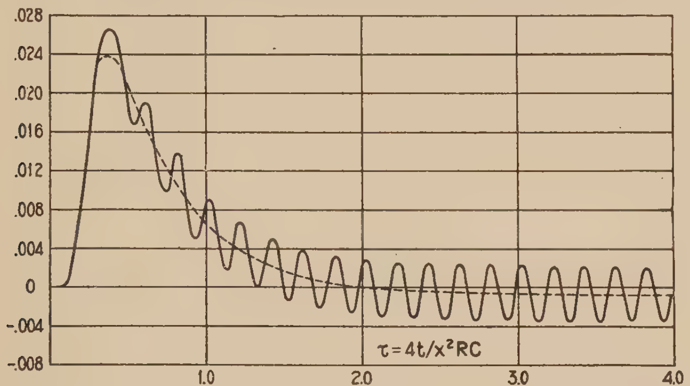

FIG. 8.—Non-inductive cable ($G=0$). Building-up of alternating current. Applied e.m.f. $\sin \omega t$; $\omega = 10\pi \dfrac{4}{x^2RC}$.

and (161). Thus if the applied e.m.f. is $\sin \omega t$, and ω is sufficiently large, the cable current is

$$\frac{2}{\sqrt{\pi}xR}\frac{1}{\omega'}\frac{d}{d\tau}\frac{e^{-1/\tau}}{\sqrt{\tau}}$$

by (160) and (170) while, if the impressed e.m.f. is $\cos \omega t$, it is

$$\frac{2}{\sqrt{\pi}xR}\left(\frac{1}{\omega'}\right)^2\frac{d^2}{d\tau^2}\frac{e^{-1/\tau}}{\sqrt{\tau}}$$

by (161) and (170). Here $\omega' = \alpha\omega/4$ and $\tau = 4t/\alpha$.

CHAPTER VII

THE PROPAGATION OF CURRENT AND VOLTAGE IN THE TRANSMISSION LINE

We now take up the more important and difficult problem of investigating the propagation phenomena in the transmission line. The transmission line has distributed series resistance R and inductance L, and distributed shunt capacity C and leakage conductance G. It is the addition of the series inductance L which makes our problem more difficult and at the same time introduces the phenomena of true propagation with finite velocity, as distinguished from the diffusion phenomena of the cable problem. The cable theory serves very well for the problems of trans-oceanic telegraphy[1] but is quite inadequate in the problems of telephonic transmission.

If I denotes the current and V the voltage at point x on the line, the well known differential equations of the problem are:

$$\left(L\frac{d}{dt}+R\right)I=-\frac{\partial}{\partial x}V, \tag{195}$$

$$\left(C\frac{d}{dt}+G\right)V=-\frac{\partial}{\partial x}I.$$

Replacing d/dt by p, these become

$$\begin{aligned}(Lp+R)\,I&=-\frac{\partial}{\partial x}V,\\ (Cp+G)V&=-\frac{\partial}{\partial x}I.\end{aligned} \tag{196}$$

From the second of these equations

$$\frac{\partial V}{\partial x}=-\frac{1}{Cp+G}\frac{\partial^2}{\partial x^2}I$$

[1] With the installation of the new submarine cable, continuously loaded with permalloy, this statement must be modified. In this cable, the inductance plays a very important part, and is responsible for the greatly increased speed of signaling obtainable. (See paper by Buckley, A. I. E. E., Aug., 1925, "The Loaded Submarine Telegraph Cable.")

and substitution in the first gives

$$(Lp+R)(Cp+G)I=\frac{\partial^2}{\partial x^2}I. \tag{197}$$

Similarly if we eliminate I, we get

$$(Lp+R)(Cp+G)V=\frac{\partial^2}{\partial x^2}V. \tag{198}$$

If we assume a solution of the form

$$V=Ae^{-\gamma x}+Be^{\gamma x}$$

where A and B are arbitrary constants, substitution shows that the solution satisfies the differential equation for V provided

$$\gamma^2=(Lp+R)(Cp+G). \tag{199}$$

From equation (196) it then follows that

$$\begin{aligned} I&=\frac{\gamma}{Lp+R}(Ae^{-\gamma x}-Be^{\gamma x}) \\ &=\frac{Cp+G}{\gamma}(Ae^{-\gamma x}-Be^{\gamma x}). \end{aligned} \tag{200}$$

Now restricting attention to the infinitely long line extending along the positive x axis, with voltage V_0 impressed directly on the line at $x=0$, the reflected wave vanishes and we get

$$\begin{aligned} V&=V_0e^{-\gamma x}, \\ I&=\frac{Cp+G}{\gamma}V_0e^{-\gamma x}, \\ \gamma^2&=(Lp+R)(Cp+G). \end{aligned} \tag{201}$$

Now let us write

$$\gamma^2=\frac{1}{v^2}[(p+\rho)^2-\sigma^2] \tag{202}$$

where

$$\begin{aligned} v&=1/\sqrt{LC}, \\ \rho&=\frac{R}{2L}+\frac{G}{2C}, \\ \sigma&=\frac{R}{2L}-\frac{G}{2C}. \end{aligned}$$

Then setting $V_0=1$, the *operational equations* of the problem become

$$V=e^{-\frac{x}{v}\sqrt{(p+\rho)^2-\sigma^2}}, \tag{203}$$

$$I=v\left(C+\frac{G}{p}\right)p\frac{e^{-\frac{x}{v}\sqrt{(p+\rho)^2-\sigma^2}}}{\sqrt{(p+\rho)^2-\sigma^2}}. \tag{204}$$

Now consider the operational equation, defining a new variable F:

$$F=p\frac{e^{-\frac{x}{v}\sqrt{(p+\rho)^2-\sigma^2}}}{\sqrt{(p+\rho)^2-\sigma^2}}. \tag{205}$$

It follows at once from our operational rules, and (203) and (204), that

$$I=v\left(C+G\int_0^t dt\right)F, \tag{206}$$

$$V=-v\int_0^t\frac{\partial F}{\partial x}dt. \tag{207}$$

Our problem is thus reduced to evaluating the function F, from the operational equation (205). This equation can be solved by aid of the operational rules and formulas already given. The process is rather complicated, and there is less chance of error if we deal instead with the integral equation of the problem

$$\frac{e^{-\frac{x}{v}\sqrt{(p+\rho)^2-\sigma^2}}}{\sqrt{(p+\rho)^2-\sigma^2}}=\int_0^\infty F(t)e^{-pt}\,dt. \tag{208}$$

Now let us search through our table of definite integrals. We do not find this integral equation as it stands, but we do observe that formula (m) resembles it, and this resemblance suggests that formula (m) can be suitably transformed to give the solution of (208). We therefore start with the formula

$$\frac{e^{-\lambda\sqrt{p^2+1}}}{\sqrt{p^2+1}}=\int_\lambda^\infty e^{-pt}J_0(\sqrt{t^2-\lambda^2})dt. \tag{m}$$

This, regarded as an integral equation, defines a function which is zero for $t<\lambda$ and has the value $J_0(\sqrt{t^2-\lambda^2})$ for $t>\lambda$, J_0 being the Bessel function of order zero. We now transform (m) as follows:

(1) Let $\lambda p=q$ and $t/\lambda=t_1$. Substituting in (m) we get

$$\frac{e^{-\sqrt{q^2+\lambda^2}}}{\sqrt{q^2+\lambda^2}}=\int_1^\infty e^{-qt_1}J_0(\lambda\sqrt{t_1{}^2-1})dt_1.$$

Now, in order to keep our original notation in p and t, replace q to p and t_1 by t; we get

$$\frac{e^{-\sqrt{p^2+\lambda^2}}}{\sqrt{p^2+\lambda^2}}=\int_1^\infty e^{-pt}J_1(\lambda\sqrt{t^2-1})dt. \tag{m.1}$$

(2) In (m.1) make the substitution $p=q+\mu$ and then in the final expression replace q by p; we get

$$\int_1^\infty e^{-pt}.e^{-\mu t}J_0(\lambda\sqrt{t^2-1})dt=\frac{e^{-\sqrt{(p+\mu)^2+\lambda^2}}}{\sqrt{(p+\mu)^2+\lambda^2}}. \tag{m.2}$$

(3) In (m.2) make the substitution $p=\frac{x}{v}q$ and $t_2=\frac{x}{v}t$, and ultimately replace q by p and t_2 by t; we get

$$\int_{x/v}^{\infty} e^{-pt}.e^{-\mu_1 t} J_0\left(\lambda_1\sqrt{t^2-\frac{x^2}{v^2}}\right)dt=\frac{e^{-\frac{x}{v}\sqrt{(p+\mu_1)^2+\lambda_1^2}}}{\sqrt{(p+\mu_1)^2+\lambda_1^2}} \qquad \text{(m.3)}$$

where $\lambda_1=\frac{v}{x}\lambda$ and $\mu_1=\frac{v}{x}\mu$. (They are, of course, as yet, arbitrary parameters, except that they are restricted to positive values.)

(4) Now if we compare (m. 3) with the integral equation (208) for F, we see that they are identical provided we set

$$\mu_1=\rho,$$
$$\lambda_1=i\sigma=\sigma\sqrt{-1}.$$

Introducing these relations, we have

$$\int_{x/v}^{\infty} e^{-pt}.e^{-\rho t} I_0\left(\sigma\sqrt{t^2-x^2/v^2}\right)dt=\frac{e^{-\frac{x}{v}\sqrt{(p+\rho)^2-\sigma^2}}}{\sqrt{(p+\rho)^2-\sigma^2}}. \qquad \text{(m.4)}$$

Here I_0 denotes the Bessel function of imaginary argument; thus $J_0(iz)=I_0(z)$.

It follows from (m.4) and the integral equation (208) that

$$\begin{aligned} F(t)&=0 \text{ for } t<x/v, \qquad \text{(209)}\\ &=e^{-\rho t} I_0(\sigma\sqrt{t^2-x^2/v^2}) \text{ for } t>x/v. \end{aligned}$$

Having now solved for $F=F(t)$, the current and voltage are gotten from equations (206) and (207). Thus

$$\begin{aligned} I&=0 \text{ for } t<x/v, \qquad \text{(210)}\\ &=\sqrt{\frac{C}{L}}F(t)+vG\int_{x/v}^{t} F(t)dt \text{ for } t\geqq x/v. \end{aligned}$$

The corresponding voltage formula is

$$\begin{aligned} V&=0 \text{ for } t<x/v,\\ &=e^{-\rho x/v}+\frac{\sigma x}{v}\int_{x/v}^{t}\frac{e^{-\rho\tau}I_1(\sigma\sqrt{\tau^2-x^2/v^2})}{\sqrt{\tau^2-x^2/v^2}}d\tau \text{ for } t>x/v. \qquad \text{(211)} \end{aligned}$$

Here $I_1(\sigma\sqrt{\tau^2-x^2/v^2})$ is the Bessel function of order 1: thus $-iJ_1(iz)=I_1(z)$. The function is entirely real. The derivation of formula (211) is a little troublesome, owing to the discontinuous character of the function F; the detailed steps are given in an appendix to this chapter.

The preceding solution depends for its outstanding directness and simplicity on the recognition of the infinite integral identity

(m), into which the integral equation of the problem can be transformed. When such identities are known their value in connection with the solution of operational equations requires no emphasis. On the other hand, we cannot always expect to find such an identity in the case of every operational equation; and, particularly in such an important case as the transmission equation it would be unfortunate to have no alternative mode of solution. Fortunately a quite direct series expansion solution is obtainable from the operational equation, and this will now be derived. As a matter of convenience we shall restrict the derivation to the voltage formula

$$V=e^{-\frac{x}{v}\sqrt{(p+\rho)^2-\sigma^2}}. \tag{203}$$

As a further matter of mere convenience we shall assume that $G=0$, so that $\sigma=\rho$ and (203) becomes

$$V=e^{-\tau\sqrt{p^2+2\rho p}} \tag{203-a}$$

where $\tau=x/v$.

The method holds equally well for the current equation (204) and for the general case $\sigma\neq\rho$.

Write (203-a) as

$$V=e^{-\tau p(1+2\rho/p)^{1/2}}$$

and expand the exponential factor $(1+2\rho/p)^{1/2}$ by the binomial theorem; thus

$$(1+2\rho/p)^{1/2}=1+\frac{\rho}{p}+\alpha_2\left(\frac{\rho}{p}\right)^2+\alpha_3\left(\frac{\rho}{p}\right)^3+\cdots$$

so that

$$V=e^{-\tau p}.e^{-\rho\tau}.\exp\left(-\frac{\alpha_2\tau\rho^2}{p}-\frac{\alpha_3\tau\rho^3}{p^2}-\frac{\alpha_4\tau\rho^4}{p^3}-\cdots\right).$$

Now the operational equation

$$v=\exp\left(-\frac{\alpha_2\tau\rho^2}{p}-\frac{\alpha_3\tau\rho^3}{p^2}-\frac{\alpha_4\tau\rho^4}{p^3}-\cdots\right)$$

can be expanded in inverse powers of p; thus

$$v=1+\frac{\beta_1}{p}+\frac{\beta_2}{p^2}+\frac{\beta_3}{p^3}+\cdots.$$

the power series solution of which is

$$v(t)=1+\frac{\beta_1 t}{1!}+\frac{\beta_2 t^2}{2!}+\frac{\beta_3 t^3}{3!}+\cdots.$$

It follows at once from the preceding and Theorem VII that

$$V(t)=0 \text{ for } t<\tau$$
$$=e^{-\rho\tau}\left(1+\beta_1\frac{(t-\tau)}{1!}+\beta_2\frac{(t-\tau)^2}{2!}+\cdots\right) \text{ for } t>\tau.$$

If the coefficients β_1, β_2 . . . are evaluated, a simple matter of elementary algebra, the foregoing expansion in the retarded time $t-\tau$ will be found to agree with the solution (211) when σ is put equal to ρ.

We shall now discuss the outstanding features of the propagation phenomena in the light of equations (210) and (211) for the current and voltage. We observe, first, that we have a true finite velocity of propagation $v=1/\sqrt{LC}$. No matter what the form of impressed e.m.f. at the beginning of the line ($x=0$), its effect does not reach the point x of the line until a time $t=x/v$ has elapsed. Consequently $v=x/t$ is the velocity with which the wave is propagated. This is a strict consequence of the distributed inductance and capacity of the line and depends only on them, since $v=1/\sqrt{LC}$. It will be recalled that in the case of the cable, where the inductance is ignored, no finite velocity of propagation exists.

The question of velocity of propagation of the wave has been the subject of considerable confusion and misinterpretation when dealing with the steady state phenomena. It seems worth while to briefly touch on this in passing.

As has been pointed out in preceding chapters, the symbolic or complex steady state formula is gotten from the operational equation by replacing the symbol p by $i\omega$ where $i=\sqrt{-1}$ and $\omega/2\pi$ is the frequency. If this is done in the operational equation (203) for the voltage, the symbolic formula is

$$V=e^{-\frac{x}{v}\sqrt{(i\omega+\rho)^2-\sigma^2}}\,e^{i\omega t}.$$

If the expression $\sqrt{(i\omega+\rho)^2-\sigma^2}$ is separated into its real and imaginary parts we get an expression of the form

$$V=e^{-\alpha x}e^{i\omega\left(t-\beta\frac{x}{v}\right)},$$

where

$$\beta=\sqrt{\frac{\omega^2+\sigma^2-\rho^2+\sqrt{(\omega^2+\sigma^2-\rho^2)^2+4\omega^2\rho^2}}{2\omega^2}}$$

and

$$\alpha=\rho/\beta v.$$

Now if we keep the expression $t-\beta\frac{x}{v}$ constant, that is, if we move along the line with velocity $dx/dt=v/\beta$, the phase of the wave will remain constant. This is interpreted often as meaning that the velocity of propagation of the wave is v/β. Now since β is greater than unity and only approaches unity as the frequency becomes indefinitely great, the inference is frequently made that the velocity of propagation depends upon and increases to a limiting value v, with the frequency. This velocity, however, is not the true velocity of propagation, which is always v, but is the *velocity of phase propagation in the steady state.* This distinction is quite important and failure to bear it in mind has led to serious mistakes.

Returning to equation (210) and (211) we see that after a time interval $t=x/v$ has elapsed since the unit e.m.f. was impressed on the cable, the voltage at point x suddenly jumps from zero to the value $e^{-\rho x/v}$ while the current correspondingly jumps to the value $\sqrt{\frac{C}{L}}e^{-\rho x/v}$. The exponential factor $\rho x/v$ is

$$x\left(\frac{R}{2L}+\frac{G}{2C}\right)\sqrt{LC}=x\left(\frac{R}{2}\sqrt{\frac{C}{L}}+\frac{G}{2}\sqrt{\frac{L}{C}}\right)=ax$$

which will be recognized as the *steady state attenuation factor* for high frequencies. Similarly $\sqrt{C/L}$ is the steady state admittance of the line for high frequencies. The sudden jumps in the current and voltage at time $t=x/v$ are called the heads of the current and voltage waves. If, instead of a unit e.m.f., a voltage $f(t)$ is impressed on the line at time $t=0$, the corresponding heads of the waves are $f(0)e^{-ax}$ and $\sqrt{C/L}\,f(0)e^{-ax}$ for voltage and current respectively. These expressions follow at once from the integral formula

$$\begin{aligned}x(t)&=\frac{d}{dt}\int_0^t f(t-\tau)h(\tau)d\tau\\&=f(0)h(t)+\int_0^t f'(t-\tau)h(\tau)d\tau.\end{aligned}$$

The tails of the waves, that is, the parts of the waves subsequent to the time $t=x/v$, are more complicated and will depend on the distance x along the line and on the line parameters ρ and σ. The two simplest cases are the *non-dissipative* line, and the *distortionless* line.

The ideal non-dissipative line, quite unrealizable in practice, is one in which both R and G are zero. In this case $\rho = \sigma = 0$, and formulas (210) and (211) become

$$\begin{aligned} I &= 0 \text{ for } t < x/v, \\ &= \sqrt{\frac{C}{L}} \text{ for } t > x/v, \\ V &= 0 \text{ for } t < x/v, \\ &= 1 \text{ for } t > x/v. \end{aligned}$$

Both current and voltage jump, at time $t = x/v$, to their steady values. If an e.m.f. $f(t)$ is impressed on the line at time $t = 0$, the corresponding current and voltage waves are

$$\begin{aligned} I &= 0 \text{ for } t < x/v, \\ &= \sqrt{\frac{C}{L}} f(t - x/v) \text{ for } t > x/v, \\ V &= 0 \text{ for } t < x/v, \\ &= f(t - x/v) \text{ for } t > x/v. \end{aligned}$$

Consequently the ideal non-dissipative line transmits the waves with finite velocity v, without attenuation or distortion. Such a line is, of course, the ideal transmission system.

The non-dissipative line is, of course, purely theoretical and unrealizable in practice; the *distortionless* line is, however, approximately realizable, and as the name implies, transmits without distortion of wave form. The distortionless line is one in which the line constants are so related that

$$\sigma = \frac{R}{2L} - \frac{G}{2C} = 0.$$

If this condition is satisfied, formulas (210) and (211) become

$$\begin{aligned} I &= 0 \text{ for } t < x/v, \\ &= \sqrt{\frac{C}{L}} e^{-ax} \text{ for } t > x/v, \\ V &= 0 \text{ for } t < x/v, \\ &= e^{-ax} \text{ for } t > x/v. \end{aligned}$$

Furthermore, if the impressed e.m.f. is $f(t)$, the corresponding current and voltage waves are:

$$\begin{aligned} I &= 0 \text{ for } t < x/v, \\ &= \sqrt{\frac{C}{L}} e^{-ax} f(t - x/v) \text{ for } t > x/v, \\ V &= 0 \text{ for } t < x/v, \\ &= e^{-ax} f(t - x/v) \text{ for } t > x/v. \end{aligned}$$

The distortionless line, therefore, transmits the waves without distortion of wave form, but attenuates the waves by the factor e^{-ax}. Such a line is an ideal transmission system as regards preservation of wave form, but introduces serious attenuation losses. For example, if a line has normally negligible leakage, and leakage is introduced to secure the condition $R/L=G/C$, the line is thereby rendered distortionless but the attenuation is doubled.

One of Heaviside's most important contributions to wire transmission theory was to point out the properties of the distortionless line, its approximately realizable character, and to base on it a correct theory of telephonic transmission.

The character of the wave propagation when the parameters ρ and σ are not restricted to special values, can only be roughly inferred from inspection of the formulas, and then only when the properties of the Bessel function I_0 and I_1 have been studied. Fortunately these functions have been computed and tabulated for small values of the argument, and have simple asymptotic expansions for large values. It is therefore a simple matter to compute and graph a representative set of curves which show the current and voltage waves for various values of ρ, σ and x. For this purpose it is convenient to introduce a change of variables and write:

$$\tau=vt$$
$$a=\rho/v$$
$$b=\sigma/v$$

whence the formulas for current and voltage become:

$$I=\sqrt{\frac{C}{L}}e^{-a\tau}I_0(b\sqrt{\tau^2-x^2})$$
$$+(a-b)\sqrt{\frac{C}{L}}\int_x^\tau e^{-a\tau}I_0(b\sqrt{\tau^2-x^2})d\tau, \tag{210-a}$$

$$V=e^{-ax}+bx\int_x^\tau\frac{e^{-a\tau}I_1(b\sqrt{\tau^2-x^2})}{\sqrt{\tau^2-x^2}}d\tau. \tag{211-a}$$

Figs. (9) to (18) give a representative set of curves illustrating the form of the propagated current and voltage waves for different lengths of line, and different values of the line parameters a and b, or ρ and σ.

The curves of Figs. (9) and (10) show the current entering the line in response to a unit e.m.f. applied at time $t=0$. The line

is assumed to be non-leaky ($b=0$) and is computed for two different values of the parameter a. We see that the current instantly jumps to the value $\sqrt{C/L}$ and then begins to die

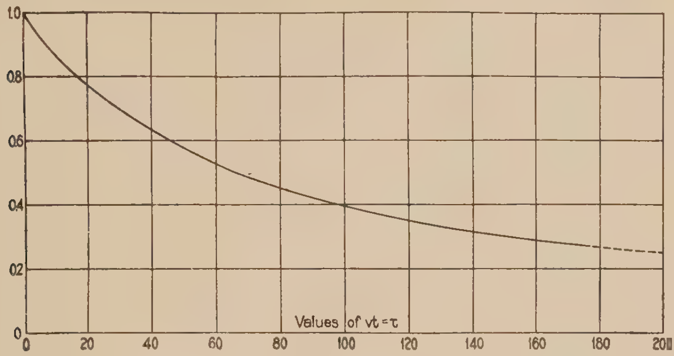

FIG. 9.—Current entering line; $\frac{R}{2}\sqrt{\frac{C}{L}}=a=0.0132$; $G=0$.
Multiply ordinates by $\sqrt{C/L}$.

away, the rate at which it dies away depending on and increasing with the parameter $a=\frac{R}{2}\sqrt{\frac{C}{L}}$.

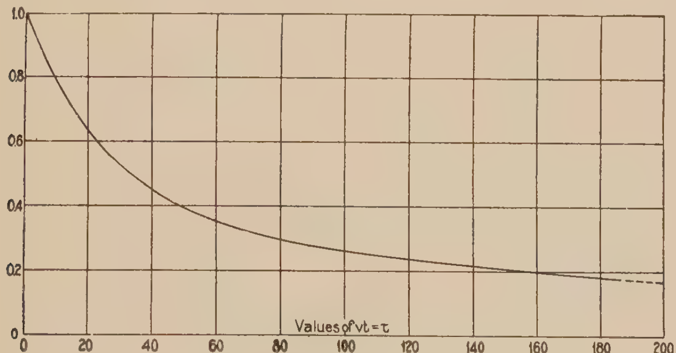

FIG. 10.—Current entering line; $\frac{R}{2}\sqrt{\frac{C}{L}}=a=0.2645$; $G=0$.
Multiply ordinates by $\sqrt{C/L}$.

If we now consider a point x out on the line, the current is zero until $\tau=x$, at which time it jumps to the value $\sqrt{C/L}\,e^{-ax}$. It then begins to die away provided x and a are such that $ax<2$. If, however, we are considering a point at which $ax>2$, the

current begins to rise instead of fall after the initial jump, and may attain a maximum value very large compared with the head

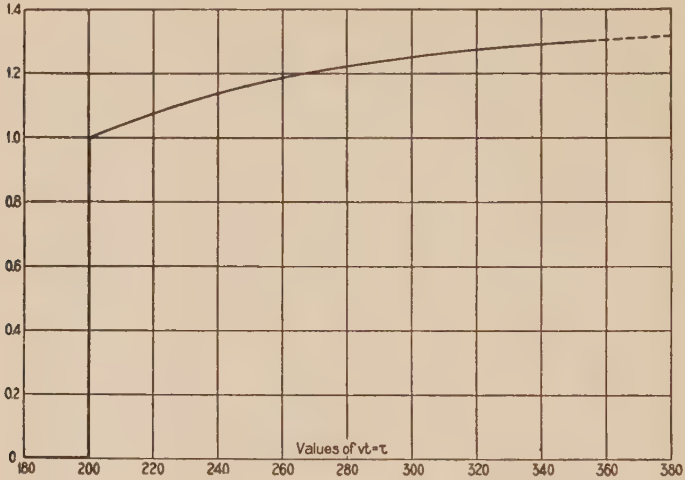

FIG. 11.—Propagated current in line; $x = 200$; $\frac{R}{2}\sqrt{\frac{C}{L}} = a = 0.0132$; $G = 0$.

Multiply ordinates by $\sqrt{C/L}.e.^{-2.64}$

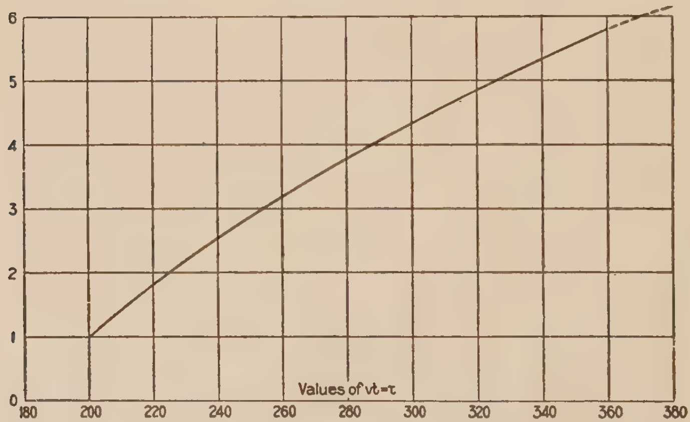

FIG. 12.—Propagated current in line; $x = 200$; $\frac{R}{2}\sqrt{\frac{C}{L}} = 0.02645$; $G = 0$.

Multiply ordinates by $\sqrt{C/L}.e^{-5.29}$.

before it starts to die away. This is shown in the curves of Figs. (11), (12) and (13), also computed for the non-leaky line ($b = 0$). From these curves we see that, as the length of the line

and the parameter a increase, the relative magnitude of the tail, as compared with the head of the wave, increases. Finally when the line becomes very long, the head of the wave becomes

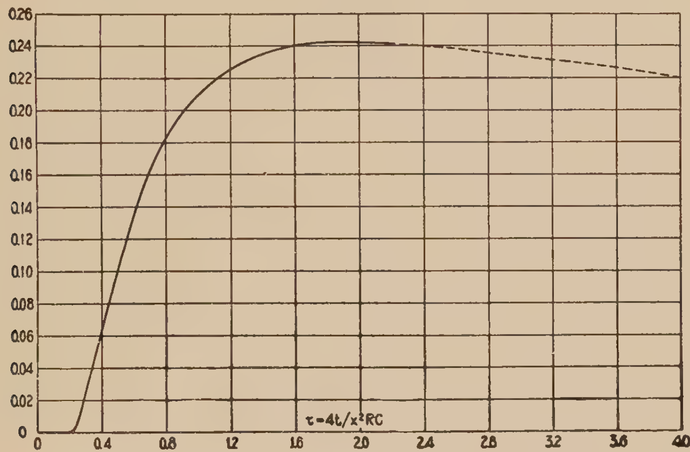

FIG. 13.—Propagated current in line; $\frac{R}{2}\sqrt{\frac{C}{L}}\, x = 10$; $G = 0$. Multiply ordinates by $2/Rx$.

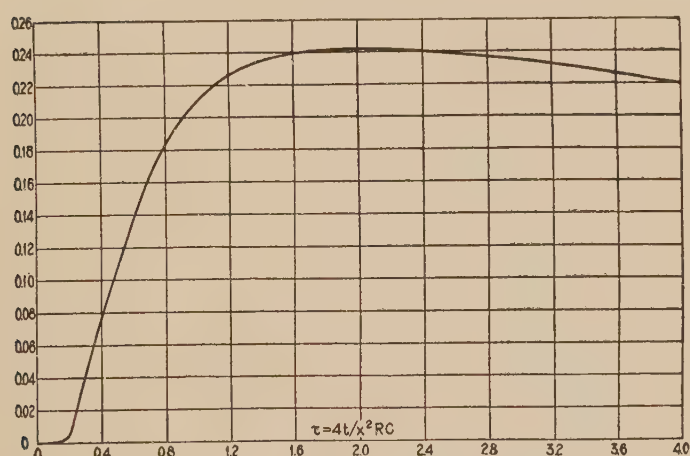

FIG. 14.—Propagated current in cable. Multiply ordinates by $2/Rx$.

negligibly small, and the wave, except in the neighborhood of its head, becomes very close to that of the corresponding non-inductive cable. This is shown in curves (13) and (14), for the

line and the corresponding cable, which are plotted to the same time scale and ordinate scale to facilitate comparison. Curve (15) shows the effect of leakage in eliminating the tail. This line is not quite distortionless but nearly so.

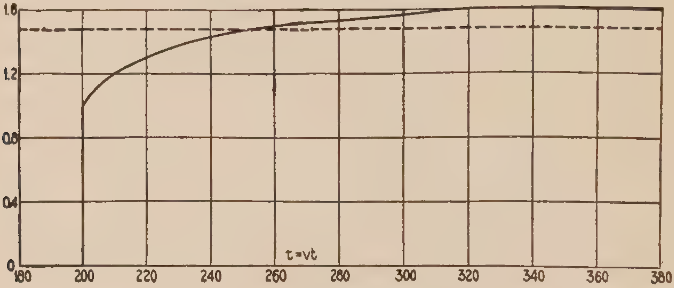

FIG. 15.—Propagated current in line; $x = 200$.

$$a = \frac{R}{2}\sqrt{\frac{C}{L}} + \frac{G}{2}\sqrt{\frac{L}{C}} = 0.0353$$

$$b = \frac{R}{2}\sqrt{\frac{C}{L}} - \frac{G}{2}\sqrt{\frac{L}{C}} = 0.01765$$

Multiply ordinates by $\sqrt{C/L}.e^{-7.06}$.

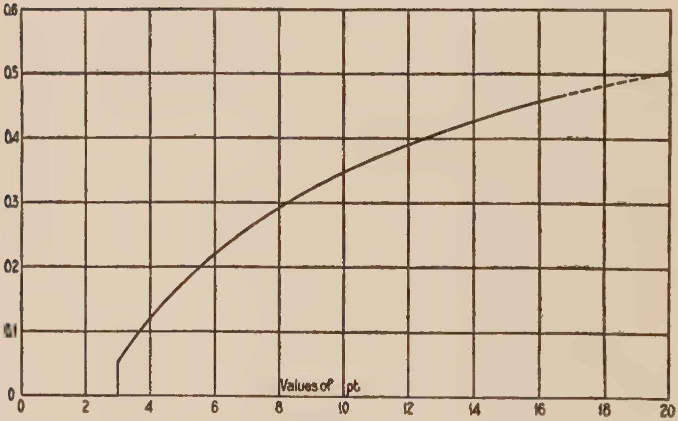

FIG. 16.—Propagated voltage in line; $\frac{R}{2}\sqrt{\frac{C}{L}}\,x = ax = 3$; $G = 0$.

An interesting feature of both current and voltage waves is that when a sufficient time has elapsed after the arrival of the head of the wave, the waves become closer and closer to the waves of the corresponding non-inductive cable; that is, to the cable having the same R, C and G. Consequently the inductance plays no part in the subsidence of the waves to their final values.

Curves (16), (17) and (18) illustrate the voltage wave for several conditions. After the arrival of the head, the wave slowly builds up to its final value. Curve (18) represents the

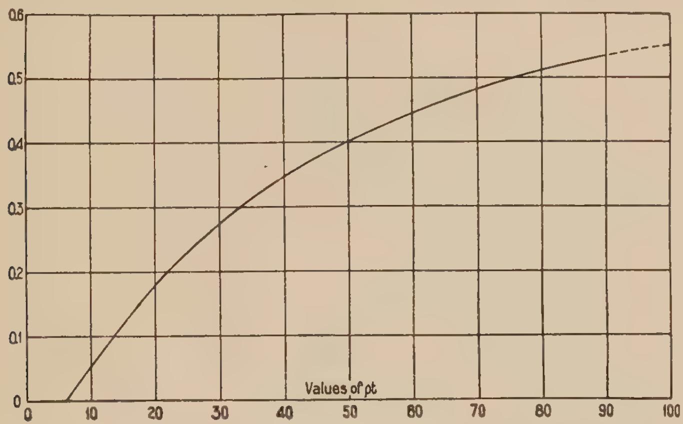

FIG. 17.—Propagated voltage in line; $\frac{R}{2}\sqrt{\frac{C}{L}}\,x = ax = 6$; $G = 0$.

case where the line is very nearly distortionless, showing how completely the distorting tail of the wave is eliminated.

So far we have confined attention to the current and voltage waves in response to a unit e.m.f. applied at time $t=0$ to the

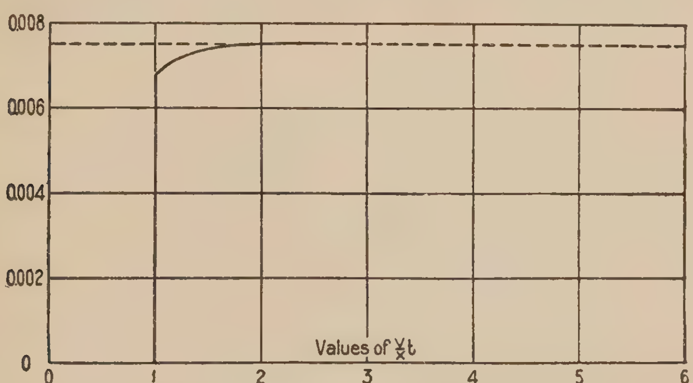

FIG. 18.—Propagated voltage in line; $\frac{R}{2}\sqrt{\frac{C}{L}}\,x = ax = 3$; $\frac{G}{2}\sqrt{\frac{L}{C}}\,x = bx = 2$.

line terminals. Of much greater technical importance is the question of the waves in response to a sinusoidal e.m.f. suddenly applied to the line terminals. In order to investigate this

important problem it is convenient to divide the expressions for the current and voltage waves as given by equations (210-a) and (211-a) into two components. We write for $\tau > x$,

$$I = \sqrt{\frac{C}{L}} e^{-ax} + J(t), \tag{210-b}$$

$$V = e^{-ax} + W(t), \tag{211-b}$$

where, by definition, $J(t)$ and $W(t)$ are the differences between the total waves and their heads. The advantage of analyzing the waves into these components is that the distortion of the waves is due to $J(t)$ and $W(t)$ respectively, while the first components of (210-b) and (211-b) introduce merely a delay. Thus, if the e.m.f. impressed at time $t=0$ is $f(t)$, the corresponding waves for $t > x/v$ or $\tau > x$, are

$$\begin{aligned} I = &\sqrt{\frac{C}{L}} e^{-ax} f(t - x/v) \\ &+ \int_{x/v}^{t} f(t - t_1) J'(t_1) dt_1, \end{aligned} \tag{212}$$

$$\begin{aligned} V = &e^{-ax} f(t - x/v) \\ &+ \int_{x/v}^{t} f(t - t_1) W'(t_1) dt_1, \end{aligned} \tag{213}$$

where $J'(t) = \frac{d}{dt} J(t)$ and $W'(t) = \frac{d}{dt} W(t)$.

The integrals of (212) and (213) can be computed and analyzed in precisely the same way as discussed in connection with the non-inductive cable problem, and are of very much the same character as the corresponding waves of the cable. In the total waves, however, as given by (212) and (213), a very essential difference is introduced by the presence of the first terms, which represent undistorted waves propagated with velocity v. Thus, if the impressed e.m.f. is $\sin \omega t$, (212) and (213) become

$$\begin{aligned} I = &\sqrt{\frac{C}{L}} e^{-ax} \sin \omega (t - x/v) \\ &+ \int_{x/v}^{t} \sin \omega (t - t_1) J'(t_1) dt_1, \text{ for } t > x/v \end{aligned} \tag{214}$$

$$\begin{aligned} V = &e^{-ax} \sin \omega (t - x/v) \\ &+ \int_{x/v}^{t} \sin \omega (t - t_1) W'(t_1) dt_1, \text{ for } t > x/v. \end{aligned} \tag{215}$$

Now the first terms of (214) and (215) are simply the usual steady state expressions for the current and voltage waves when

the frequency is sufficiently high to make the steady state attenuation constant equal to a and the phase velocity equal to v. Furthermore the integral terms become smaller and smaller as the applied frequency $\omega/2\pi$ is increased. It follows, therefore, that for high frequencies the waves assume substantially their final steady value at time $t=x/v$, and that the tails of the waves, or the transient distortion, becomes negligible. This is a consequence entirely of the presence of inductance in the line, and shows its extreme importance in the propagation of alternating waves and the reduction of transient distortion.

It should be pointed out, however, that if the line is very long and the attenuation is very high, the integral terms of (214) and (215) are not negligible unless the applied frequency is correspondingly very high. For example, on a long submarine cable, the a.c. attenuation is so large that the first terms of (214) and (215) are very small, and $J(t)$ is very large compared with $\sqrt{C/L}\ e^{-ax}$. Consequently here there is very serious transient distortion and alternating currents are therefore not adapted for submarine telegraph signalling.

This discussion may possibly be made a little clearer, without detailed analysis, if we recall the discussion of alternating current propagation in the non-inductive cable of the preceding chapter. From that analysis it follows that, when the applied frequency $\omega/2\pi$ is sufficiently high, the integral term of (214) becomes approximately

$$\frac{1}{\omega}J'(t)$$

and the complete current wave is

$$\sqrt{\frac{C}{L}}e^{-ax}\sin\omega(t-x/v)+\frac{1}{\omega}J'(t) \tag{216}$$

and similarly the voltage wave is

$$e^{-ax}\sin\omega(t-x/v)+\frac{1}{\omega}W'(t). \tag{217}$$

Now if the total attenuation ax is large the last terms of (216) and (217), before they ultimately die away, may become very large compared with the first terms, which represent the ultimate steady state.

APPENDIX TO CHAPTER VII

Derivation of Formula (211).—The only troublesome question involved in deriving (211) from (207) and (209) is that we have to

differentiate with respect to x, in accordance with (207), the discontinuous function $F(t)$. To accomplish this we write (209) in the form

$$F(t)=\phi(t-x/v)e^{-\rho t}I_0(\sigma\sqrt{t^2-x^2/v^2}) \tag{209–a}$$

where $\phi(t)$ is defined as a function which is zero for $t<x/v$ and unity for $t>x/v$. Clearly this is equivalent to (209) and permits us to deal with $F(t)$ as a *continuous* function. Now, in accordance with (207), perform the operation of differentiation upon (209–a): we get

$$-v\frac{\partial F}{\partial x}=\frac{\partial}{\partial t}\phi(t-x/v)e^{-\rho t}I_0(\sigma\sqrt{t^2-x^2/v^2})$$
$$-v\phi(t-x/v)\frac{\partial}{\partial x}e^{-\rho t}I_0(\sigma\sqrt{t^2-x^2/v^2}).$$

The first expression follows from the fact that

$$\frac{\partial}{\partial x}\phi(t-x/v)=-\frac{1}{v}\frac{\partial}{\partial t}\phi(t-x/v).$$

We observe also that $\frac{\partial}{\partial t}\phi(t-x/v)=0$ except at $t=x/v$, when it is infinite. We also observe that, for $t>x/v$,

$$\int_0^t\frac{\partial}{\partial t}\phi(t-x/v)dt=1$$

and that the whole contribution to the integral occurs at $t=x/v$. With these points clearly in mind, the expression

$$V=-v\int_0^t\frac{\partial F}{\partial x}dt$$

reduces to (211) without difficulty.

CHAPTER VIII

PROPAGATION OF CURRENT AND VOLTAGE IN ARTIFICIAL LINES AND WAVE FILTERS

The artificial line here considered is a periodic structure, composed of a series of sections connected in tandem, each section consisting of a lumped impedance z_1 in series with the line, and a lumped impedance z_2 in shunt across the line. In the artificial line which we shall consider it will be assumed that the voltage is applied at the middle of the initial or zeroth section, as shown in Fig. 19. This termination is chosen because of its practical importance, and because also of the fact that the mathematical analysis is simplified thereby. Furthermore any other termination can be regarded and dealt with as an additional terminal impedance, so there is no essential loss of generality involved.

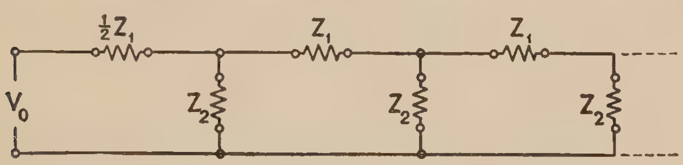

Fig. 19.

A study of the properties of the artificial line is of practical importance for several reasons:

1. The artificial line is often used as a model of an actual transmission line and it is therefore of importance to determine theoretically the degree of correspondence between the two.

2. The solution for the corresponding transmission line with continuously distributed constants is derivable from the solution for the artificial line by keeping the total inductance, resistance, capacity and leakage constant or finite, and letting the number of sections approach infinity.

3. The artificial line is very closely related, in its properties and performance, to the periodically loaded line, and its solution is, to a first approximation, a working solution for the loaded line.

4. The structure is of great importance in its own right, and when the impedance elements are properly chosen, constitutes a "wave filter."

We shall now derive the operational and symbolic equations which formulate the propagation phenomena in the artificial line. Let I_n denote the mesh current in the n^{th} section of the line; I_{n-1} the mesh current in the $(n-1)^{th}$ section etc. Now write down the expression for the voltage drop in the n^{th} section; in accordance with Kirchhoff's law we get:

$$(z_1+2z_2)I_n-z_2(I_{n-1}+I_{n+1})=0 \tag{218}$$

where, of course, the impedances have the usual significance.

Now this is a *difference* equation, as distinguished from a differential equation, but the method of solution is essentially the same. We *assume* a solution of the form

$$I_n=Ae^{-n\Gamma}+Be^{n\Gamma} \tag{219}$$

where A, B and Γ are independent of n, and substitute in (218). After some simple rearrangements we get

$$\{(z_1+2z_2)-2z_2\cosh\Gamma\}\cdot\{Ae^{-n\Gamma}+Be^{n\Gamma}\}=0 \tag{220}$$

Equation (218) is clearly satisfied by the assumed form of solution, and furthermore leaves the constants A and B arbitrary and at our disposal to satisfy any boundary conditions, provided Γ is so chosen that

$$\begin{aligned}\cosh\Gamma&=\frac{z_1+2z_2}{2z_2}\\&=1+2\rho\end{aligned} \tag{221}$$

where $\rho=z_1/4z_2$.

Now by reference to equation (219) it is easily seen that Γ is the *propagation constant* of the artificial line, precisely analogous to the propagation constant γ of the smooth line. In terms of the impedances z_1 and z_2, the propagation constant of the artificial line is determined by (221). This equation may either be regarded as an operational equation or a symbolic equation, depending on whether the impedances are expressed in terms of the operator p or in terms of $i\omega$, where ω is 2π times the frequency.

Now suppose in (221) we write $e^{\Gamma}=x$; the equation becomes

$$x+1/x=2(1+2\rho)$$

and solving for x we get

$$\begin{aligned} x=e^{\Gamma}&=(1+2\rho)+\sqrt{(1+2\rho)^2-1} \\ &=(\sqrt{1+\rho}+\sqrt{\rho})^2=(\sqrt{1+\rho}-\sqrt{\rho})^{-2}, \end{aligned} \tag{222}$$

which is an explicit formula for Γ.

Now return to equation (219) and let us assume that the line is either infinitely long, or, what amounts to the same thing, that it is closed by an impedance which suppresses the reflected wave. We assume also that a voltage V_0 is impressed at mid-series position of the zeroth section ($n=0$). Equation (219) becomes

$$I_n=Ae^{-n\Gamma}$$

and the currents in the zeroth and 1st sections are

$$I_0=A, \qquad I_1=Ae^{-\Gamma}.$$

Now, by direct application of Kirchhoff's law to the zeroth section, we have

$$V_0=(\tfrac{1}{2}z_1+z_2)I_0-z_2I_1,$$

whence

$$A\{\tfrac{1}{2}z_1+z_2(1-e^{-\Gamma})\}=V_0. \tag{223}$$

But

$$I_0=A=\frac{1}{K}V_0,$$

$$I_n=\frac{V_0}{K}e^{-n\Gamma},$$

where K is the *characteristic impedance* of the artificial line (at mid-series position). Hence by (223) and (222)

$$\begin{aligned} \frac{1}{K}&=\frac{1}{z_2}\,\frac{1}{(1-e^{-\Gamma})+2\rho} \\ &=\frac{1}{2z_2}\,\frac{1}{\sqrt{\rho+\rho^2}}=\frac{1}{\sqrt{z_1z_2}}\,\frac{1}{\sqrt{1+\rho}}. \end{aligned} \tag{224}$$

By aid of the preceding the direct current wave can be written as

$$I_n=\frac{V_0}{\sqrt{z_1z_2}}\,\frac{[\sqrt{1+\rho}-\sqrt{\rho}]^{2n}}{\sqrt{1+\rho}}. \tag{225}$$

This formula is not so physically suggestive as its equivalent

$$I_n=\frac{V_0}{K}e^{-n\Gamma}$$

but is useful when we come to the solution of the operational equation.

Before proceeding with the operational equation, and the investigation of transient phenomena in artificial lines, it will be of interest to deduce from the foregoing the unique and remarkable properties of wave filters in the steady state. For this purpose we return to equation (221)

$$\cosh \Gamma = 1 + 2\rho.$$

Now suppose that the series impedance z_1 is an inductance L and the shunt impedance z_2 a capacity C, so that, symbolically,

$$z_1 = i\omega L, \quad z_2 = \frac{1}{i\omega C}, \quad \rho = -\frac{\omega^2 LC}{4}.$$

and

$$\cosh \Gamma = 1 - \tfrac{1}{2}\omega^2 LC. \tag{226}$$

Now let us write $\Gamma = i\theta$, where $i = \sqrt{-1}$; the preceding equation becomes

$$\cos \theta = 1 - \tfrac{1}{2}\omega^2 LC \tag{227}$$

and the *ratio of currents* in adjacent sections is $e^{-i\theta}$. *Consequently if θ is a real quantity the ratio of the absolute values of the currents in adjacent sections is unity, and the current is propagated without attenuation.*

Inspection of equation (227) shows that θ is real provided the right hand side lies between $+1$ and -1; or that ω lies between 0 and $2/\sqrt{LC}$. Consequently this type of artificial line transmits, in the steady state, sinusoidal currents of all frequencies from zero to $1/\pi\sqrt{LC}$ without attenuation. It is known as the low pass filter.

If we invert the structure, that is, make the series impedance z_1 a capacity C and the shunt impedance z_2 an inductance L, so that

$$z_1 = \frac{1}{i\omega C}, \quad z_2 = i\omega L, \quad \rho = -\frac{1}{4\omega^2 LC},$$

we get, corresponding to (226) and (227),

$$\cosh \Gamma = 1 - \frac{1}{2\omega^2 LC}, \tag{228}$$

$$\cos \theta = 1 - \frac{1}{2\omega^2 LC}. \tag{228-a}$$

This type of artificial line transmits without attenuation currents of all frequencies for which the right hand side of (228-a) lies

between $+1$ and -1; that is, all frequencies from infinity to a lower limiting frequency $1/4\pi\sqrt{LC}$, while it attenuates all frequencies below this range. It is known, on this account, as the high pass filter.

It is possible by using more complicated impedances to design filters which transmit a series of bands of frequencies. We cannot, however, go into the complicated theory of wave-filters here, which has been covered in a series of important papers.[1] One point should be noted, however: transmission without attenuation implies that the impedance elements are non-dissipative. Actually, of course, all the elements introduce some loss, so that in practice the filter attenuates all frequencies. Careful design, however, keeps the attenuation very low in the transmission bands.

We shall now derive the indicial admittance formulas for some representative types of artificial lines and wave-filters from the operational formula

$$A_n=\frac{1}{\sqrt{(1+\rho)z_1z_2}}[\sqrt{1+\rho}+\sqrt{\rho}]^{-2n}. \tag{229}$$

This equation follows directly from (225) on putting $V_0=1$.

We start with the so-called low pass filter on account of its simplicity and also its great importance in technical applications. This type of filter consists of series inductance L and shunt capacity C. The general case which includes series resistance R and shunt leakage G has been worked out (see Transient Oscillations, Trans. A.I.E.E., 1919). The solution is, however, extremely complicated and will not be dealt with here. We shall, instead, consider the important and illuminating case where the series and shunt losses are so related as to make the circuit quasi-distortionless. We therefore take, operationally,

$$\begin{aligned} z_1&=pL+R=L(p+\lambda) \\ 1/z_2&=pC+G=C(p+\lambda) \end{aligned} \tag{230}$$

where $\lambda=R/L=G/C$.

We then have

$$\begin{aligned} z_1z_2&=L/C, \\ z_1/z_2&=LC(p+\lambda)^2, \\ \rho&=\frac{LC}{4}(p+\lambda)^2. \end{aligned} \tag{231}$$

[1] See Bibliography.

Now by reference to formula (229) we see that A_n is a function of $(p+\lambda)$; thus

$$A_n=\frac{1}{Z_n(p+\lambda)}=\left(1+\frac{\lambda}{p}\right)\frac{p}{(p+\lambda)Z_n(p+\lambda)}.$$

Now write

$$A_n^o=\frac{1}{Z_n(p)}.$$

It follows at once from reference to theorem V that

$$A_n=\left(1+\lambda\int_0^t dt\right)A_n^o\, e^{-\lambda t} \tag{232}$$

so that the problem is reduced to the solution of the operational equation for A_n^o. Writing $\omega_c=2/\sqrt{LC}$, we have

$$A_n^o=\sqrt{\frac{C}{L}}\frac{1}{\sqrt{1+(p/\omega_c)^2}}\left[\sqrt{1+(p/\omega_c)^2}+p/\omega_c\right]^{-2n} \tag{233}$$

$$=\sqrt{\frac{C}{L}}\frac{\omega_c}{\sqrt{p^2+\omega_c^2}}\left[\frac{\sqrt{p^2+\omega_c^2}-p}{\omega_c}\right]^{2n}.$$

Now refer to formula (n) of the table of integrals; writing $\sqrt{L/C}=k$, we see by Theorem II that

$$A_n^o=\frac{1}{k}\int_0^{\omega_c t} J_{2n}(\tau)d\tau \tag{234}$$

where $J_{2n}(\tau)$ is the Bessel function of order $2n$ and argument τ. We note also that this is the indicial admittance of the non-dissipative low pass wave-filter; that is, the current in the n'^h section in response to a unit e.m.f. applied to the initial section $(n=0)$. From (232) and (234) it follows at once that

$$A_n=e^{-\lambda t}\frac{1}{k}\int_0^{\omega_c t} J_{2n}(\tau)d\tau$$

$$+\frac{\lambda}{k}\int^t d\tau e^{-\lambda\tau}\int_0^{\omega_c\tau} J_{2n}(\tau_1)d\tau_1.$$

Integrating the second member by parts and noting that $A_n^o\,(0)=0$, this reduces to

$$A_n=\frac{1}{k}\int_0^{\omega_c t} e^{-\frac{\lambda}{\omega_c}\tau} J_{2n}(\tau)d\tau \tag{235}$$

which is the indicial admittance formula for the quasi-distortionless low pass filter, or artificial line.

Before discussing these formulas, it is of interest to derive the formula for A_n° by power series expansion. Formula (233) can be written

$$A_n^o=\frac{1}{k}\left(\frac{\omega_c}{p}\right)^{2n+1}\frac{1}{\sqrt{1+(\omega_c/p)^2}}\frac{1}{[1+\sqrt{1+(\omega_c/p)^2}]^{2n}}.$$

This can be expanded in a series in inverse powers of p; thus

$$A_n^o=\frac{1}{k2^{2n}}\left\{\left(\frac{\omega_c}{p}\right)^{2n+1}-\frac{2n+2}{2^2 1!}\left(\frac{\omega_c}{p}\right)^{2n+3}+\frac{(2n+3)\ (2n+4)}{2^4 2!}\left(\frac{\omega_c}{p}\right)^{2n+5}-\cdots\cdot\right\}.$$

Replacing $1/p^n$ by $t^n/n!$ in accordance with the Heaviside Rule we get

$$A_n^o=\frac{2}{k}\left\{\frac{1}{(2n+1)!}\left(\frac{\omega_c t}{2}\right)^{2n+1}-\frac{2n+2}{1!(2n+3)!}\left(\frac{\omega_c t}{2}\right)^{2n+3}+\frac{(2n+3)\ (2n+4)}{2!(2n+5)!}\left(\frac{\omega_c t}{2}\right)^{2n+5}-\cdots\cdots\right\}. \tag{235-a}$$

This can be recognized as the power series expansion of (234).

The *artificial cable* is also of interest and practical importance. In this structure the series impedance is a resistance R and the shunt impedance is a capacity C, so that

$$\begin{aligned}&z_1=R,\ 1/z_2=pC,\\ &z_1z_2=R/pC,\ z_1/z_2=pRC,\\ &\rho=pRC/4.\end{aligned} \tag{236}$$

Now let us return to formula (229), and expand in inverse powers of p: we get

$$A_n=\frac{1}{2^{2n}\sqrt{\rho z_1 z_2}}\left\{\frac{1}{p^n}-\frac{2n+2}{2^2 1!}\frac{1}{p^{n+1}}+\frac{(2n+3)(2n+4)}{2^4 2!}\frac{1}{p^{n+2}}-\cdots\right\}. \tag{237}$$

Now since $\sqrt{\rho z_1 z_2}=\frac{R}{4}$, we have

$$A_n=\frac{2}{2^n R}\left\{\left(\frac{2}{RCp}\right)^n-\frac{2n+2}{2.1!}\left(\frac{2}{RCp}\right)^{n+1}+\frac{(2n+3)(2n+4)}{2^2 2!}\left(\frac{2}{RCp}\right)^{n+2}-\cdots\cdot\right\}.$$

Replacing $1/p^n$ by $t^n/n!$ we get finally

$$A_n = \frac{2}{2^n R}\left\{\frac{1}{n!}\left(\frac{2t}{RC}\right)^n - \frac{(2n+2)}{2.1!(n+1)!}\left(\frac{2t}{RC}\right)^{n+1} + \frac{(2n+3)(2n+4)}{2^2.2!(n+2)!}\left(\frac{2t}{RC}\right)^{n+2} - \cdots\right\}. \tag{238}$$

For large values of n and t this series is difficult to compute or interpret. It can, however, be recognized as the series expansion of the function

$$A_n = \frac{2}{R}e^{-\frac{2t}{RC}} I_n\left(\frac{2t}{RC}\right) \tag{239}$$

where $I_n(2t/RC)$ is the Bessel function I of order n and argument $(2t/RC)$. This solution, it may be remarked, can be derived directly by a modification of the integral formula (n).

It is beyond the scope of this paper to consider other types of artificial lines and wave-filters; for a fairly extensive discussion the reader is referred to "Transient Osillations in Electric Wave-Filters," B. S. T. J., July, 1923. The low pass wave-filter, however, both in its own right and on account of its close relation to the periodically loaded line, deserves further discussion.

For the non-dissipative low pass wave-filter, we have

$$A_n^{\circ} = \frac{1}{k}\int_0^{\omega_c t} J_{2n}(\tau)d\tau \tag{234}$$

while for the quasi-distortionless low pass wave-filter

$$A_n = \frac{1}{k}\int_0^{\omega_c t} e^{-\mu\tau} J_{2n}(\tau)d\tau \tag{235}$$

where $\mu = \lambda/\omega_c = R/L\omega_c = R/2vL$.

Computation and analysis of these formulas involve an elementary knowledge of Bessel functions. The properties necessary for our purposes are briefly discussed in an appendix to this chapter.

The indicial admittances for the non-dissipative low pass filter, that is, the current in response to a steady unit e.m.f. applied at time $t=0$, are shown in the curves of Figs. 20, 21 and 22, for the initial or zeroth, the 3rd and the 5th sections, respectively. These curves together with the exact and approximate formulas given above are sufficient to give a reasonably comprehensive idea of the general character of these oscillations and their dependence on the number of sections and the constants of the filter.

It will be observed that the current is small until a time approximately equal to $2n/\omega_c = n\sqrt{L_1C_2}$ has elapsed after the voltage is applied. Consequently the low pass filter behaves as though cur-

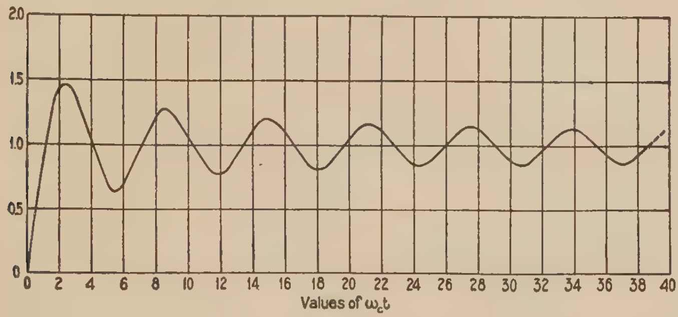

Fig. 20.—Low pass wave filter. Indicial admittance of initial section ($n = 0$). Multiply ordinates by $\sqrt{C/L}$.

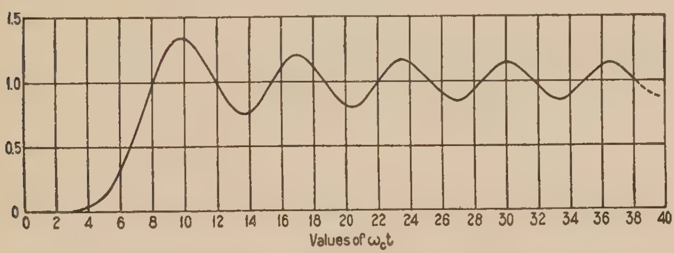

Fig. 21.—Low pass wave filter. Indicial admittance of third section ($n = 2$). Multiply ordinates by $\sqrt{C/L}$.

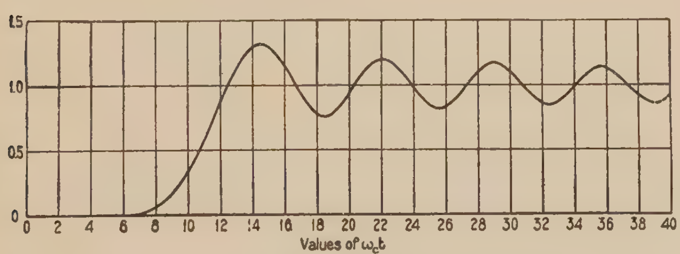

Fig. 22.—Low pass wave filter. Indicial admittance of fifth section ($n = 4$). Multiply ordinates by $\sqrt{C/L}$.

rents were transmitted with a finite velocity of propagation $\omega_c/2 = 1/\sqrt{LC}$ sections per second. This velocity is, however, only apparent or virtual since in every section the currents are actually finite for all values of time > 0.

After time $t = n\sqrt{LC}$ has elapsed the current oscillates about the value $1/k$ with increasing frequency and diminishing amplitude. The amplitude of these oscillations is approximately

$$\frac{1/k}{\sqrt{1-(2n/\omega_c t)^2}}\sqrt{\frac{2}{\pi\omega_c t}}$$

and their instantaneous frequency (measured by intervals between zeros)

$$\frac{\omega_c}{2\pi}\sqrt{1-(2n/\omega_c t)^2}.$$

The oscillations are therefore ultimately of cut-off or critical frequency $\omega_c/2\pi$ in all sections, but this frequency is approached more and more slowly as the number of filter sections is increased.

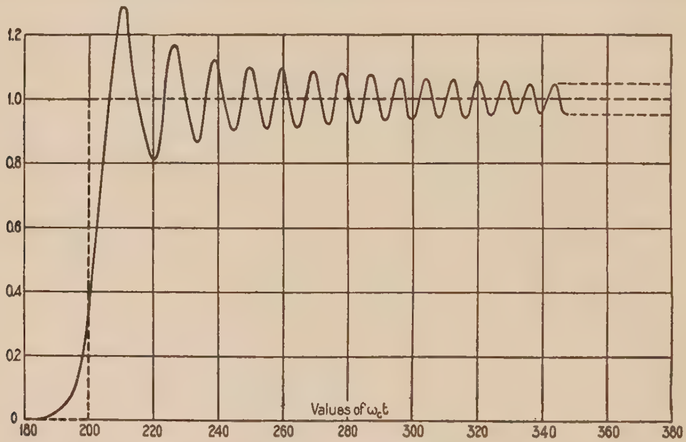

FIG. 23.—Low pass wave filter. Indicial admittance of 100th section ($n = 99$). Multiply ordinates by $\sqrt{C/L}$.

Figs. 23, 24, 25, give the indicial admittance in the 100th, 500th and 1000th section of the low-pass wave-filter. The filter itself seldom embodies more than 5 sections. The case of a large number of sections is of interest, however, because it represents a first approximation to the periodically loaded line. While the non-dissipative line is ideal and unrealizable, its study is of practical importance because in this type of line the effect of the discontinuous character of the loading of the periodically loaded line is isolated and exhibited in the clearest possible manner.

The dotted curves represent the current in the corresponding smooth line. For the smooth line, the current, as we have seen, is

discontinuous, being identically zero for a time $vt=n$ and having an instantaneous jump to its final value $\sqrt{C/L}$ at $vt=n$. The current in the artificial or periodically loaded line differs from that

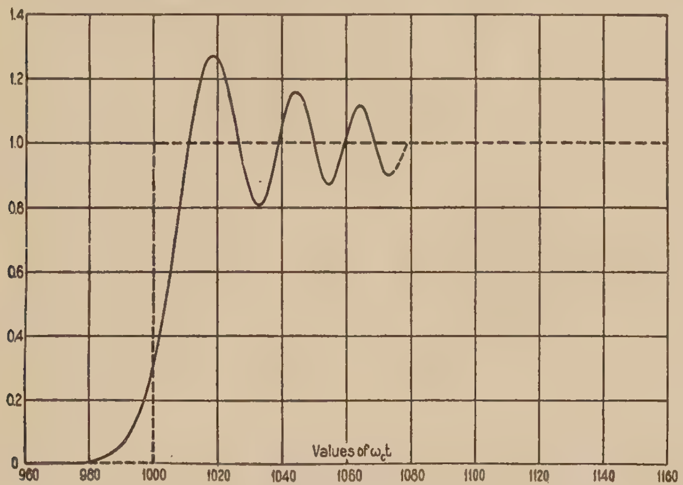

FIG. 24.—Low pass wave filter. Indicial admittance of 500th section ($n=499$) Multiply ordinates by $\sqrt{C/L}$.

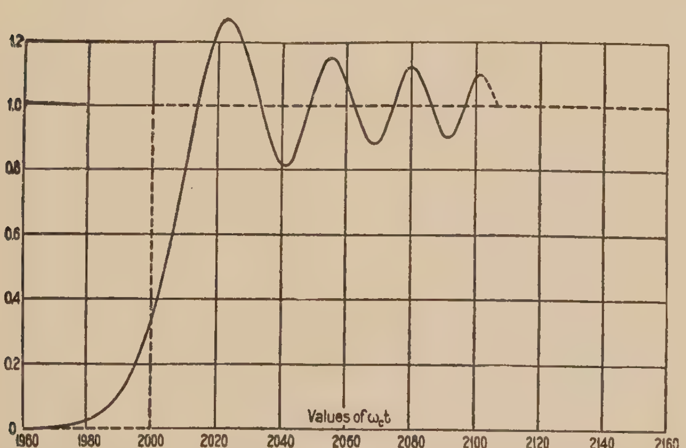

FIG. 25.—Low pass wave filter. Indicial admittance of 1000th section ($n=999$). Multiply ordinates by $\sqrt{C/L}$.

in the corresponding smooth line in three important respects: (1) the absence of the abrupt discontinuous wave front, (2) the presence of superposed oscillations, and (3) *the absence of a true*

finite velocity of propagation. It will be observed, however, that the current in any section is negligibly small or even sensibly zero until $vt=n$, so that the current is propagated with a *virtual* velocity $1/\sqrt{LC}$ per section. The presence of a well marked wave front is also evident although this is not abrupt, as in the smooth line. The effective slope of the wave front becomes smaller as the current wave travels out on the line, decreasing noticeably as the number of sections is increased. When the number of sections becomes large, however, the decrease in the slope is not rapid, being in the 500th section about 60 per cent. of that in the 100th section.

The superposed oscillations are of interest. These are initially of a frequency depending upon and decreasing with the number of sections, n, but in all sections ultimately attaining the frequency

$$\frac{1}{\pi\sqrt{LC}}=\frac{v}{\pi}$$

which is the critical or cut-off frequency of the line, above which steady state currents are attenuated during transmission and below which they are unattenuated. When vt is large compared with n the amplitude of these oscillations becomes $\sqrt{1/\pi vt}$ so that they ultimately die away and the current approaches the value $\sqrt{C/L}$ for all sections. The current in the loaded line is thus asymptotic to the current in the corresponding smooth line and oscillates about it with diminishing amplitude and increasing frequency.

Since the abscissas of these curves represent values of $2vt=2t/\sqrt{LC}$, and the ordinates are to be multiplied by $\sqrt{C/L}$ to translate into actual values, the curves are of universal application for all values of the constants L and C.

The investigation of the building up of alternating currents in wave filters and loaded lines is very important. It depends for the non-dissipative case on the properties of the definite integrals

$$\int_0^{\omega_c t}\sin w\tau\, J_n(\tau)d\tau,$$

$$\int_0^{\omega_c t}\cos w\tau\, J_n(\tau)d\tau.$$

where $w=\omega/\omega_c$ and $\omega=2\pi$ times the applied frequency. The mathematical discussion is, however, quite complicated and will

not be entered into here. The reader, who wishes to follow this further, is referred to Transient Oscillations, Trans. A. I. E. E., 1919 and Transient Oscillations in Electric Wave Filters, B. S. T. J., July, 1923.

APPENDIX TO CHAPTER VIII

Note on Bessel Functions.—The Bessel Functions of the first kind, $J_n(x)$ and $I_n(x)$, are defined, when n is zero or a positive integer, by the absolutely convergent series

$$J_n(x)=\frac{x^n}{2^n.n!}\left\{1-\frac{x^2}{2(2n+2)}+\frac{x^4}{2.4(2n+2)(2n+4)}\right.$$
$$\left.-\frac{x^6}{2.4.6(2n+2)(2n+4)(2n+6)}+\cdots\right\},$$

$$I_n(x)=\frac{x^n}{2^n.n!}\left\{1+\frac{x^2}{2(2n+2)}+\frac{x^4}{2.4(2n+2)(2n+4)}\right.$$
$$\left.+\frac{x^6}{2.4.6.(2n+2)(2n+4)(2n+6)}+\cdots\right\}.$$

In the following discussion of the properties of these functions it will be assumed that the argument x is a pure real quantity.

For large values of the argument (x large compared with n), the behavior of the functions is shown by the asymptotic expansions:

$$I_n(x)=\frac{e^x}{\sqrt{2\pi x}}\left\{1-\frac{4n^2-1}{1!(8x)}+\frac{(4n^2-1)(4n^2-9)}{2!(8x)^2}\right.$$
$$\left.-\frac{(4n^2-1)(4n^2-9)(4n^2-25)}{3!\,(8x)^3}+\cdots\right\},$$

$$J_n(x)=\sqrt{\frac{2}{\pi x}}\left\{P_n\cos\left(x-\frac{2n+1}{4}\pi\right)-Q_n\sin\left(x-\frac{2n+1}{4}\pi\right)\right\},$$

where

$$P_n=1-\frac{(4n^2-1)(4n^2-9)}{2!\,(8x)^2}$$
$$+\frac{(4n^2-1)(4n^2-9)(4n^2-25)(4n^2-49)}{4!\,(8x)^4}-\cdots,$$

$$Q_n=\frac{4n^2-1}{8x}-\frac{(4n^2-1)(4n^2-9)(4n^2-25)}{3!(8x)^3}+\cdots.$$

We thus see that I_n increases indefinitely and behaves ultimately as

$$\frac{e^x}{\sqrt{2x\pi}}.$$

The function $J_n(x)$, however, is oscillatory and ultimately behaves as

$$\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{2n+1}{4}\pi\right).$$

For all orders of n

$$\int_0^\infty J_n(x)dx=1.$$

The properties of $J_n(x)$ may be described qualitatively as follows:—

When the argument is less than the order $(0\leqq x<n)$ the function is very small and positive, and is initially zero (except when $n=0$). In the neighborhood of $x=n$, the function begins to build up and reaches a maximum a little beyond the point $x=n$. Thereafter the function oscillates with increasing frequency and diminishing amplitude, and ultimately behaves as

$$\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{2n+1}{4}\pi\right).$$

When $n=0$, the initial value is unity, but the subsequent behavior of the function is as described above.

A more precise description of the function is gotten from the following approximate formulas.[1]

$$J_n(x)\doteq B_n(x)\cos\Omega_n(x),\quad \text{for } x>n$$

where

$$B_n(x)=\sqrt{\frac{2}{\pi x}}\frac{1}{\left(1-\frac{m^2}{x^2}+\frac{3}{2}\frac{m^2}{x^4}\frac{1}{(1-m^2/x^2)^2}\right)^{1/4}},$$

$$\Omega_n(x)=x\left[\sqrt{1-\frac{m^2}{x^2}}+\frac{m}{x}\sin^{-1}\left(\frac{m}{x}\right)-\frac{m^2}{4x^4}\frac{1}{(1-m^2/x^2)^{3/2}}\right]-\frac{2n+1}{4}\pi,$$

$$\Omega_n'(x)=\frac{d}{dx}\Omega_n(x),$$

$$=\sqrt{1-\frac{m^2}{x^2}+\frac{3}{2}\frac{m^2}{x^4}\frac{1}{(1-m^2/x^2)^2}},$$

[1] These formulas are due to Mr. Zobel (See "Transient Oscillations in Electric Wave-filters" Bell System Technical Journal, July, 1923.) Somewhat similar formulas had previously been developed by Graf and Gubler (Einleitung in die Theorie der Besselschen Funktionen), and by Nicholson (Phil. Mag., 1910, p. 249).

and

$$m^2 = n^2 - 1/4.$$

This approximate formula is valid only where $x > n$, its accuracy increasing with x and with n. For all orders of n it is quite accurate beyond the first zero of the function.

The "instantaneous frequency" of oscillation is approximately

$$\frac{1}{2\pi}\Omega_n'(x) = \frac{1}{2\pi}\sqrt{1 - \frac{m^2}{x^2} + \frac{3}{2}\frac{m^2}{x^4}\frac{1}{(1-m^2/x^2)^2}}.$$

By this it is meant that at any point $x(x > n)$ the interval between successive zeros is approximately $\pi/\Omega'(x)$. Otherwise stated, in the neighborhood of any point x, the function behaves like a sinusoid of amplitude $B_n(x)$ and frequency $\omega/2\pi$ where $\omega = \Omega'_n(x)$.

The following approximate formulas, while not sufficiently precise for the purpose of accurate computation except for quite large values of x, clearly exhibit the character of the functions for values of the argument $x > n$, and of the order $n > 2$.

$$J_n(x) \doteq h_n\sqrt{\frac{2}{\pi x}}\cos(q_n x - \theta_n),$$

$$J_n'(x) = -q_n h_n\sqrt{\frac{2}{\pi x}}\sin(q_n x - \theta_n),$$

$$\int_0^x J_n(x)dx = 1 + \frac{h_n}{q}\sqrt{\frac{2}{\pi x}}\sin(q_n x - \theta_n),$$

where

$$h_n = \left(\frac{1}{1-n^2/x^2}\right)^{1/4} = 1 + \frac{n^2}{4x^2},$$

$$q_n = \sqrt{1 - n^2/x^2},$$

$$\theta_n = \frac{2n+1}{4}\pi - n\sin^{-1}(n/x).$$

CHAPTER IX

THE FINITE LINE WITH TERMINAL IMPEDANCES

So far in our discussions of wave propagation in lines and wave-filters, we have confined attention to the case where the impressed voltage is applied directly to the infinitely long line. We have found that, by virtue of this restriction, the indicial admittance functions of the important types of transmission systems are rather easily derived and expressible in terms of well known functions, and the essential phenomena of wave propagation clearly exhibited. In practice, however, we are concerned with lines of finite length with the voltage impressed on the line through a terminal impedance Z_1 and the distant end closed by a second terminal impedance Z_2. We now take up the problem presented by such a system.

Let $K=K(p)$ denote the characteristic operational impedance of the line, and $\gamma=\gamma(p)$ the operational propagation constant of the line. We have then

$$\begin{aligned} V &= Ae^{-\gamma x}+Be^{\gamma x}, \\ I &= \frac{1}{K}Ae^{-\gamma x}-\frac{1}{K}Be^{\gamma x}, \end{aligned} \tag{240}$$

where A and B are so far arbitrary constants. To determine these constants we assume an e.m.f. E impressed on the line at $x=0$ through a terminal impedance Z_1 and the line closed at $x=s$ by a second terminal impedance Z_2. At $x=s$ we have therefore

$$Z_2 I = V$$

whence from (240)

$$\frac{Z_2}{K}e^{-\gamma s}A-\frac{Z_2}{K}e^{\gamma s}B=Ae^{-\gamma s}+Be^{\gamma s}$$

and

$$B=-\frac{1-\rho_2}{1+\rho_2}e^{-2\gamma s}A \tag{241}$$

where $\rho_2=Z_2/K$.

At $x=0$ we have

$$V=E-Z_1 I$$

whence

$$A+B=E-\frac{Z_1}{K}A+\frac{Z_1}{K}B,$$

$$(1+\rho_1)A+(1-\rho_1)B=E, \tag{242}$$

where $\rho_1=Z_1/K$.

From (241) and (242) we get

$$A=\frac{1+\rho_2}{(1+\rho_1)(1+\rho_2)-(1-\rho_1)(1-\rho_2)e^{-2\gamma s}}E$$

$$B=\frac{-(1-\rho_2)e^{-2\gamma s}}{(1+\rho_1)(1+\rho_2)-(1-\rho_1)(1-\rho_2)e^{-2\gamma s}}E$$

and finally

$$I_x=\frac{E}{K+Z_1}\frac{e^{-\gamma x}+\frac{1-\rho_2}{1+\rho_2}e^{-\gamma(2s-x)}}{1-\frac{1-\rho_1}{1+\rho_1}\frac{1-\rho_2}{1+\rho_2}e^{-2\gamma s}}. \tag{243}$$

If we replace E by a unit e.m.f. we get the operational formula for the indicial admittance A_x; thus

$$A_x=\frac{\lambda}{K}\frac{e^{-\gamma x}+\mu_2 e^{-\gamma(2s-x)}}{1-\mu_1\mu_2 e^{-2\gamma s}}=\frac{1}{Z_x(p)} \tag{244}$$

where

$$\lambda=K/(K+Z_1),$$

$$\mu_1=\frac{1-\rho_1}{1+\rho_1}=\frac{K-Z_1}{K+Z_1},$$

$$\mu_2=\frac{1-\rho_2}{1+\rho_2}=\frac{K-Z_2}{K+Z_2}.$$

K,γ,Z_1,Z_2,μ_1 and μ_2 are, of course, functions of the operator p.

The integral equation corresponding to the operational formula (244) is

$$\frac{1}{pZ_x(p)}=\int_0^\infty e^{-pt}A_x(t)dt. \tag{245}$$

Now by (244) we can expand $1/Z_x(p)$; it is

$$\begin{aligned}\frac{1}{Z_x(p)}&=\lambda\frac{e^{-\gamma x}}{K}+\lambda\mu_2\frac{e^{-\gamma(2s-x)}}{K}\\&+\lambda\mu_1\mu_2\frac{e^{-\gamma(2s+x)}}{K}+\lambda\mu_1\mu_2^2\frac{e^{-\gamma(4s-x)}}{K}\\&+\lambda\mu_1^2\mu_2^2\frac{e^{-\gamma(4s+x)}}{K}+\cdots\end{aligned} \tag{246}$$

Now we observe that $e^{-\gamma x}/K$ is simply the operational formula for the indicial admittance at point x of an infinitely long line with unit e.m.f. impressed directly on the line at $x=0$. This will be denoted by $a_x(t)$. Similarly $e^{-\gamma(2s-x)}/K$ is the operational formula for the indicial admittance at point $(2s-x)$ with unit e.m.f. impressed directly on the line at $x=0$. This will be denoted by $a_{2s-x}(t)$, etc.

Recognition of this fact allows us to derive a formal solution in terms of a series of reflected waves. For let a set of functions $v_0, v_1, v_2, v_3, \ldots$ satisfy and be defined by the operational equations

$$\begin{aligned} v_0 &= \lambda(p) = \lambda \\ v_1 &= \lambda\mu_2 \\ v_2 &= \lambda\mu_1\mu_2 \\ v_3 &= \lambda\mu_1\mu_2^2, \text{ etc.} \end{aligned} \qquad (247)$$

It then follows from the preceding and theorem II that

$$A_x(t) = \frac{d}{dt}\int_0^t d\tau \left\{ \begin{array}{r} v_0(t-\tau)a_x(\tau) + v_1(t-\tau)a_{2s-x}(\tau) \\ + v_2(t-\tau)a_{2s+x}(\tau) + \cdots \end{array} \right\}. \qquad (248)$$

If, therefore, we know the indicial admittance of the infinitely long line with unit e.m.f. directly applied and if we can solve the operational equations (247), then $A_x(t)$ is given by (248) by integration. This solution may well present formidable difficulty in the way of computation. It is, however, formally straightforward and the numerical computation is entirely possible, the only question being as to whether the importance of the problem justifies the necessary expenditure of time and effort. Without any computations, however, the solution (248) admits of considerable instructive interpretation by inspection. The first term represents the current at point x of an infinitely long line in response to a unit e.m.f. impressed at $x=0$ through an impedance Z_1; $v_0 = v_0(t)$ is the corresponding voltage across the line terminals proper. The second term is a reflected wave from the other terminal due to the terminal irregularity which exists there. The third term is a reflected wave from the sending end terminal, etc. The solution is therefore a wave solution and is expanded in a form which corresponds exactly with the sequence of the phenomena, which it represents.

The solution takes a particularly instructive form when $Z_1 = k_1 K$ and $Z_2 = k_2 K$ where k_1 and k_2 are numerics. Then

$$\begin{aligned} v_0 &= \frac{1}{1+k_1} \\ v_1 &= \frac{1}{1+k_1}\frac{1-k_2}{1+k_2} \\ v_2 &= \frac{1}{1+k_1}\frac{1-k_2}{1+k_2}\frac{1-k_1}{1+k_1}, \text{ etc.} \end{aligned} \tag{249}$$

and

$$A_x(t) = \frac{1}{1+k_1}\left\{ \begin{array}{l} a_x(t) + \dfrac{1-k_2}{1+k_2} a_{2s-x}(t) \\ + \dfrac{1-k_1}{1+k_1}\dfrac{1-k_2}{1+k_2} a_{2s+x}(t) + \cdots \end{array} \right\}. \tag{250}$$

If $k_1 = 0$, $k_2 = 1$ we have the case of the e.m.f. impressed directly on the sending end of the line and the distant end closed through its characteristic impedance; the solution reduces to

$$A_x(t) = a_x(t)$$

as, of course, it should by definition.

If $k_1 = 0$ and $k_2 = \infty$, we have the case of the line open-circuited at the distant end, and the solution reduces to

$$A_x(t) = \left\{ a_x(t) - a_{2s-x}(t) - a_{2s+x}(t) + a_{4s-x}(t) + \cdots \right\}. \tag{251}$$

Finally, if both k_1 and k_2 are zero, the line is shorted and

$$A_x(t) = \left\{ a_x(t) + a_{2s-x}(t) + a_{2s+x}(t) + a_{4s-x}(t) + \cdots \right\}. \tag{252}$$

The operational equations (247) admit of further interesting and instructive physical interpretation without computation. Consider a circuit consisting of an impedance Z_1 in series with an impedance K. Let a unit e.m.f. be applied to this circuit and let v_0 be the resultant voltage across the impedance K. Then, operationally,

$$v_0 = \frac{K}{K+Z_1} = \lambda$$

so that v_0, thus defined in physical terms, is the v_0 of equations (247).

Now let this voltage be impressed on a circuit consisting of an impedance $2Z_2$ in series with an impedance $K - Z_2$ so that the total impedance is $K + Z_2$. Let the resultant voltage drop across

the impedance element $K-Z_2$ be denoted by v_1; then operationally

$$v_1=\frac{K}{K+Z_1}\cdot\frac{K-Z_2}{K+Z_2}=\lambda\mu_2$$

which agrees with v_1 as given by equation (247).

Similarly if voltage v_1 is applied to a circuit consisting of an impedance $2Z_1$ in series with an impedance $K-Z_1$ and if v_2 denote the voltage drop across impedance $K-Z_1$, then

$$v_2=\lambda\mu_1\mu_2$$

We can thus see physically what the voltages $v_0, v_1, v_2, \ldots$ mean in terms of simple circuits consisting of K and Z_1 in series and K and Z_2 in series respectively.

I shall now work out a specific problem exemplifying the preceding theory. The example is made as simple as possible for two reasons. First because its simplicity makes it more instructive than when the phenomena depicted and the essentials of the mathematical methods are obscured by complicated formulas and extensive computations. Secondly while the general method of solution illustrated is thoroughly practical we cannot hope to arrive at the numerical solutions of the complicated problems without a large amount of laborious computations. Problems involving transmission lines with complicated terminal impedances are among the most difficult, as regards actual numerical solution, of any which present themselves in mathematical physics. On the other hand, the formal solution (248) gives at a glance the essential character of the phenomena involved.

The specific problem we shall deal with may be stated as follows: A unit e.m.f. is directly impressed on the terminals of a transmission line of length s, the distant end of which is closed by a condenser C_0. The line is supposed to be non-dissipative, its constants being inductance L and capacity C per unit length. Required the current at any point $x (x<s)$ of the line.

We write $\sqrt{L/C}=k$, $1/\sqrt{LC}=v$: then by virtue of the preceding analysis of transmission line propagation the indicial admittance a_x of the *infinitely long line* is given by

$$\begin{aligned} a_x&=0, \text{ for } t<x/v,\\ &=\frac{1}{k}, \text{ for } t>x/v. \end{aligned}$$

The operational characteristic impedance is, of course, $k=\sqrt{L/C}$, and the terminal impedances Z_1 and Z_2 are given by

$$Z_1=0,$$
$$Z_2=1/pC_0.$$

Referring now to equation (244) we have:

$$\lambda=1,\ \mu_1=1,$$
$$\mu_2=\frac{k-1/pC_0}{k+1/pC_0}=\frac{kC_0p-1}{kC_0p+1}.$$

Consequently, referring to equations (247), we have, operationally,

$$v_0=1$$
$$v_1=v_2=\frac{kC_0p-1}{kC_0p+1}$$
$$v_3=v_4=\left(\frac{kC_0p-1}{kC_0p+1}\right)^2$$
$$v_5=v_6=\left(\frac{kC_0p-1}{kC_0p+1}\right)^3.$$

In order to determine these functions we have therefore to solve the general operational equation

$$V_n=\left(\frac{kC_0p-1}{kC_0p+1}\right)^n$$

where V_n denotes either v_{2n-1} or v_{2n}.

In order to eliminate the coefficient kC_0 we make use of theorem VI, and write

$$\phi_n=\left(\frac{p-1}{p+1}\right)^n.$$

In accordance with that theorem

$$V_n(t)=\phi_n(t/kC_0).$$

We therefore start with the operational equation

$$\phi_n=\left(\frac{p-1}{p+1}\right)^n.$$

Now the solution of this operational equation is very easy and can be expressed in a number of ways. We require it expressed in the

form most easily computed. The following appears best adapted for our purposes. Consider the auxiliary operational equation:

$$\sigma_n = \left(\frac{p-2}{p}\right)^n$$

$$= \left(p^n - 2\frac{n}{1!}p^{n-1} + 2^2\frac{(n)(n-1)}{2!}p^{n-2} + \cdots + (-1)^n 2^n\right)\frac{1}{p^n}.$$

The explicit solution is gotten by placing $1/p^n$ by $t^n/n!$ and p^n by d^n/dt^n, whence

$$\sigma_n(t) = \left(\frac{d^n}{dt^n} - 2\frac{n}{1!}\frac{d^{n-1}}{dt^{n-1}} + \cdots + (-1)^n 2^n\right)\frac{t^n}{n!},$$

$$= 1 - \frac{n}{1!}\frac{2t}{1!} + \frac{n(n-1)}{2!}\frac{(2t)^2}{2!} + \cdots + (-1)^n\frac{(2t)^n}{n!}.$$

But writing

$$\sigma_n = \left(\frac{p-2}{p}\right)^n = \frac{1}{H(p)}$$

it follows that

$$\phi_n = \left(\frac{p-1}{p+1}\right)^n = \frac{1}{H(p+1)}$$

$$= \frac{p+1}{p}\cdot\frac{p}{p+1}\frac{1}{H(p+1)}$$

$$= \left(1+\frac{1}{p}\right)\cdot\frac{p}{p+1}\frac{1}{H(p+1)}.$$

Referring now to theorem V, we see that

$$\phi_n(t) = \left(1 + \int_0^t dt\right)\sigma_n(t)\cdot e^{-t}.$$

Since we have already solved for $\sigma_n(t)$, this determines $\phi_n(t)$ and hence $V_n(t)$. The functions v_0, v_1, v_2 . . . are therefore determined.

Now refer back to equation (248) giving the required current in terms of $v_0, {}_1v_2$ . . . and the admittances $a_x(t), a_{2s-x}(t_n)$, . . . It follows at once by substitution of the preceding that

$$A_x(t) = \frac{1}{k}\left\{v_0\left(t-\frac{x}{v}\right) + v_1\left(t-\frac{2s-x}{v}\right) + v_2\left(t-\frac{2s+x}{v}\right) + \cdots\right\}$$

the functions v_0, v_1, v_2 being zero for negative values of the argument. This result may possibly require a little explanation.

Consider the expression

$$\frac{d}{dt}\int_0^t f(t-\tau)\cdot 1(\tau)d\tau$$

where $1(t)$ denotes a function which is zero for $t<t_0$ and unity for $t>t_0$. It is evidently identical with the admittance $a_x(t)$ provided the proper value is assigned to t_0.

Now since $1(t)=0$ for $t<t_0$ and unity for $t>t_0$ the preceding may be written as zero for $t<t_0$ and

$$\frac{1}{k}\frac{d}{dt}\int_{t_0}^t f(t-\tau)d\tau \qquad \text{for } t>t_0$$

which is equal to $f(t-t_0)$.

If we set $x=0$, we get the current entering the line; thus

$$\begin{aligned} A_0(t)=\frac{1}{k}\Bigg\{ & v_0(t)+v_1\left(t-\frac{2s}{v}\right)+v_2\left(t-\frac{2s}{v}\right) \\ & +v_3\left(t-\frac{4s}{v}\right)+v_4\left(t-\frac{4s}{v}\right)+\cdots\Bigg\} \\ =\frac{1}{k}\Bigg\{ & 1+2V_1\left(t-\frac{2s}{v}\right)+2V_3\left(t-\frac{4s}{v}\right) \\ & +2V_5\left(t-\frac{6s}{v}\right)+\cdots\cdot\Bigg\}. \end{aligned}$$

This has been computed for the case where $\sqrt{sLC_0}=10$ and is shown in Fig. 26. Referring to this figure we see that the current

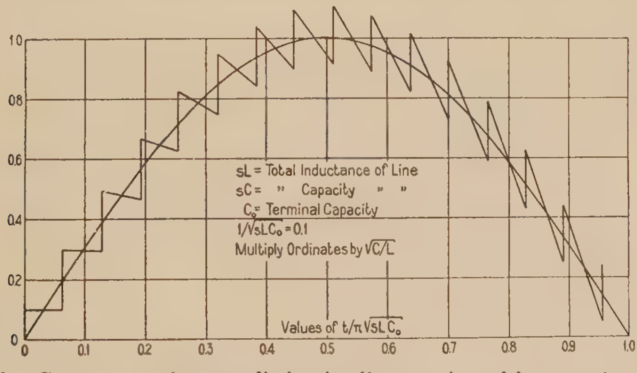

FIG. 26.—Current entering non-dissipative line terminated by capacity C_0 unit e.m.f. applied to line.

jumps at $t=0$ to the value $\sqrt{C/L}=1/k$, and keeps this constant value for a time interval $2s/v$. At this instant the first reflected

wave arrives and the current takes another jump of $2/k$. Thereafter it begins to decrease very slowly until time $t=4s/v$ at which time it takes another jump of $2/k$. Thereafter we have a series of jumps of $2/k$ at time intervals $2s/v$, the current decreasing between successive jumps. The smooth curve is the indicial admittance of an oscillation circuit consisting of an inductance sL in series with a capacity C_0. We see, therefore, that the current in the line oscillates with discontinuous jumps about the current in the corresponding oscillation circuit. Since the whole circuit contains no resistance, the oscillations never die away, but continue to oscillate, as shown, about the curve

$$\sqrt{\frac{C}{L}}\sin\left(\frac{t}{\sqrt{sLC_0}}\right)$$

which is the indicial admittance of the corresponding oscillation circuit.

I shall now discuss a method of solving circuit theory problems, quite generally applicable to complicated networks, and particu-

FIG. 27.

larly useful in dealing with transmission lines terminated in impedances. I have found it particularly useful in arriving at numerical solutions where other methods prove far more laborious. It is also of mathematical interest, as it applies another type of integral equation to the problems of electric circuit theory.

Suppose that we have a network with two sets of terminals as shown in Fig. 27.[1] Now suppose that terminals 22 are short circuited and a unit e.m.f. inserted between terminals 11. Let the resultant current flowing between terminals 11 be denoted by $S_{11}(t)=S_{11}$ and that between terminals 22 by $S_{21}(t)=S_{21}$. S_{11} is the driving point indicial admittance with respect to terminals 11 and S_{21} the transfer indicial admittance of terminals 22 with respect to 11 under short circuit conditions.

Similarly if terminals 11 are shortcircuited and a unit e.m.f. inserted between terminals 22 the current flowing between termi-

[1] Regarding conventions as to signs, see the Appendix to this chapter.

nals 22 is denoted by $S_{22}(t) = S_{22}$ and that flowing between terminals 11 by $S_{12}(t) = S_{12}$. If the network is passive, i.e., contains no internal source of energy, it follows from the reciprocal theorem that $S_{21} = S_{12}$. As far as the two sets of terminals are concerned, the network is completely specified by the indicial admittances $S_{11}, S_{22}, S_{21} = S_{12}$.

Now let a voltage $V_1(t) = V_1$ be inserted between terminals 11, and a voltage $V_2(t) = V_2$ between terminals 22. The current flowing between terminals 11, denoted by I_1 is

$$I_1(t) = \frac{d}{dt}\int_0^t V_1(\tau)S_{11}(t-\tau)d\tau + \frac{d}{dt}\int_0^t V_2(\tau)S_{12}(t-\tau)d\tau \qquad (253)$$

while the corresponding current between terminals 22 is

$$I_2(t) = \frac{d}{dt}\int_0^t V_2(\tau)S_{22}(t-\tau)d\tau + \frac{d}{dt}\int_0^t V_1(\tau)S_{21}(t-\tau)d\tau. \qquad (254)$$

Now consider two networks of indicial admittances $S_{11}, S_{22}, S_{12} = S_{21}$ and $T_{11}, T_{22}, T_{12} = T_{21}$ respectively and let them be connected

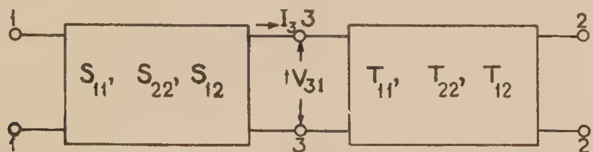

FIG. 28.

in tandem as shown in Fig. 28 to form a compound network. We require the indicial admittances of the compound network in terms of the indicial admittances of the component networks.

Short circuit terminals 22 of the compound network and insert a unit e.m.f. between terminals 11. Let $V_{31}(t)$ denote the resultant voltage between terminals 33 measured in the direction of the arrow, and I_3 the current flowing between the networks. We have then the two following expressions for the current I_3.

$$I_3 = S_{21}(t) - \frac{d}{dt}\int_0^t V_{31}(\tau)S_{22}(t-\tau)d\tau \qquad (255)$$

and

$$I_3 = \frac{d}{dt}\int_0^t V_{31}(\tau)T_{11}(t-\tau)d\tau. \qquad (256)$$

Equating, we get

$$\frac{d}{dt}\int_0^t V_{31}(\tau)[S_{22}(t-\tau) + T_{11}(t-\tau)]d\tau = S_{21}(t). \qquad (257)$$

By precisely similar reasoning, if terminals 11 are short circuited and a unit e.m.f. inserted between terminals 22, and the corresponding voltage across terminals 33 denoted by V_{32}, we have[1]

$$\frac{d}{dt}\int_0^t V_{32}(\tau)[S_{22}(t-\tau)+T_{11}(t-\tau)]d\tau = T_{12}(t). \tag{258}$$

Equations (257) and (258) are integral equations of the Poisson type which completely determine V_{31} and V_{32} in terms of the indicial admittances S and T. We shall discuss the solution of these equations presently.

If $U_{11}, U_{22}, U_{21} = U_{12}$ denote the indicial admittances of the compound network we have at once

$$U_{11} = S_{11}(t) - \frac{d}{dt}\int_0^t V_{31}(\tau)S_{12}(t-\tau)d\tau \tag{259}$$

$$U_{22} = T_{22}(t) - \frac{d}{dt}\int_0^t V_{32}(\tau)T_{21}(t-\tau)d\tau \tag{260}$$

and

$$\begin{aligned} U_{21} = U_{12} &= \frac{d}{dt}\int_0^t V_{31}(\tau)T_{21}(t-\tau)d\tau \\ &= \frac{d}{dt}\int^t V_{32}(\tau)S_{12}(t-\tau)d\tau. \end{aligned} \tag{261}$$

If, therefore, equations (257) and (258) are solved for V_{31} and V_{32}, the required indicial admittances of the compound network are given by (259), (260) and (261) in terms of the indicial admittances of the component networks.

A simple example will now be worked out illustrating the method of solution just discussed. Suppose that a unit e.m.f. is impressed on a transmission line (infinitely long) of distributed constants R,L,C, through a terminal resistance R_0. Required the terminal line voltage V.

The operational equation of this problem is gotten in the usual manner. The current entering the line is

$$V\sqrt{\frac{Cp}{Lp+R}}.$$

[1] V_{32} being opposite to V_{31} in direction.

It is also obviously equal to $\frac{1}{R_0}(1-V)$: equating the two expressions, and rearranging we get:

$$V=\frac{\sqrt{\frac{Lp+R}{Cp}}}{R_0+\sqrt{\frac{Lp+R}{Cp}}}.$$

Writing $R/2L=\rho$ and setting $R_0=\sqrt{L/C}$, this becomes

$$V=\frac{\sqrt{1+2\rho/p}}{1+\sqrt{1+2\rho/p}}. \tag{262}$$

This operational equation can, of course, be solved in a number of ways, though, as a matter of fact, its numerical solution is quite troublesome. This point will be returned to later; we shall first formulate the problem in accordance with the method just discussed.

The indicial admittance of the line is known; it is

$$\sqrt{\frac{C}{L}}e^{-\rho t}I_0(\rho t)=A(t).$$

Consequently the current entering the line is explicitly

$$\frac{d}{dt}\int_0^t V(\tau)A(t-\tau)d\tau.$$

But the current is also equal to $\frac{1}{R_0}(1-V(t))$; equating, we get

$$V(t)=1-R_0\frac{d}{dt}\int_0^t V(\tau)A(t-\tau)d\tau.$$

Performing the indicated differentiations

$$V(t)=1-R_0A(0)V(t)-R_0\int_0^t V(\tau)A'(t-\tau)d\tau.$$

Now $A(0)=\sqrt{\frac{C}{L}}$ and

$$A'(t)=\rho e^{-\rho t}[I_1(\rho t)-I_0(\rho t)]\sqrt{\frac{C}{L}}$$

and $R_0=\sqrt{L/C}$; therefore the equation becomes

$$V(t)=\frac{1}{2}+\frac{\rho}{2}\int_0^t V(t-\tau)[I_0(\rho\tau)-I_1(\rho\tau)]e^{-\rho\tau}d\tau.$$

As a matter of convenience we change the time scale to ρt, and get

$$V(t)=\frac{1}{2}+\frac{1}{2}\int_0^t V(t-\tau)[I_0(\tau)-I_1(\tau)]e^{-\tau}d\tau \tag{263}$$
$$=\frac{1}{2}-\frac{1}{2}\int_0^t d\tau V(t-\tau)\frac{d}{d\tau}e^{-\tau}I_0(\tau)$$

where it is understood that t is actually ρt. This is the integral equation of the problem and is in the canonical form of Poisson's integral equation.

Before solving this equation numerically I shall show how a simple approximate solution is obtainable immediately; an advantage often attaching to this type of integral equation.

The function $\frac{d}{dt}e^{-t}I_0(t)$ is equal to -1 for $t=0$ and converges rapidly to zero. $V(t)$ has, as we know from the operational equation, the initial value 1/2 and the final value 1. Neither function changes sign. It follows from the mean value theorem that the equation can be written as

$$V(t)=\frac{1}{2}-\frac{1}{2}V(t)\int_0^{\alpha t}\frac{d}{dt}e^{-t}I_0(t)dt$$

where $\alpha \leqq 1$. Integrating

$$V(t)=\frac{1}{2}-\frac{1}{2}V(t)[e^{-\alpha t}I_0(\alpha t)-1]$$

and

$$V(t)=\frac{1}{1+e^{-\alpha t}I_0(\alpha t)}. \tag{264}$$

The correct initial and final values of $V(t)$ result for all final values of $\alpha \leqq 1$; so that approximately

$$V(t)=\frac{1}{1+e^{-t}I_0(t)}.$$

This equation, while not exact, except for $t=0$ and t very large, shows faithfully the general character of $V(t)$ and the way it approaches its final value unity. For large values of t

$$e^{-t}I_0(t)=1/\sqrt{2\pi t}$$

whence

$$V(t)=\frac{1}{1+1/\sqrt{2\pi t}}, \quad t\geqq 8. \tag{264-a}$$

Approximations of the foregoing type are not always possible and may not be of sufficient accuracy. I shall therefore give next a method of numerical solution which is generally applicable to integral equations of this type and works quite well in practice. We shall write the integral equation in the more general form

$$u(x)=f(x)+\int_0^x u(x-y)k(y)dy \qquad (265)$$

where $f(x)$ and $k(y)$ are known and $u(x)$ unknown. The method depends on the numerical integration of the definite integral. Let us divide the x scale into small intervals d and for convenience write

$$u(nd)=u_n$$
$$f(nd)=f_n$$
$$k(nd)=k_n.$$

Now from the integral equation we have at once

$$u(0)=u_0=f_0$$
$$u(d)=u_1=f_1+\int_0^d u(d-y)k(y)dy.$$

Now if d is taken sufficiently small

$$\int_0^d u(d-y)k(y)dy=\frac{d}{2}[u_1k_0+u_0k_1]$$

whence

$$u_1=f_1+\frac{d}{2}[u_1k_0+u_0k_1]$$

and

$$u_1=\frac{1}{1-k_0d/2}[f_1+u_0k_1d/2]$$

which determines u_1 since u_0 is known. Similarly

$$u_2=f_2+d[\tfrac{1}{2}u_0k_2+u_1k_1+\tfrac{1}{2}u_2k_0]$$

which determines u_2. Proceeding in the same manner

$$u_3=f_3+d[\tfrac{1}{2}u_0k_3+u_1k_2+u_2k_1+\tfrac{1}{2}u_3k_0], \text{ etc.}$$

In this way we determine the value of $u(x)$, point by point from the recurrence formula

$$u_n=\frac{f_n+d[\frac{1}{2}u_0k_n+u_1k_{n-1}+u_2k_{n-2}+\cdot\cdot+u_{n-1}k_1]}{1-\frac{1}{2}k_0d}. \qquad (266)$$

The result of the application of numerical integration, in accordance with formula (266), to the integral equation (263) is shown in Fig. (29). The dotted curve is a plot of the approxi-

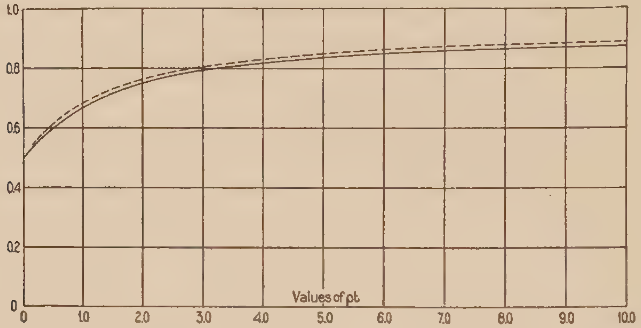

Fig. 29.—Line terminal voltage unit e.m.f. impressed on line through resistance. $R_0 = \sqrt{L/C}$.

mate solution as given by equation (264), for $\alpha = 1$. We see that the voltage starts with the value 1/2 and slowly reaches its ultimate value, unity, its approach to unity, for large values of t, being in accordance with the formula

$$V(t) = \frac{1}{1 + 1/\sqrt{2\pi t}}.$$

The application of the foregoing method to the transmission line problem proceeds as follows. Let $S_{11}(t)$, $S_{22}(t)$ and $S_{12}(t)$ be the short indicial admittances of the line. $S_{11}(t)$ is the current entering the line (at $x=0$) with unit e.m.f. directly impressed and the distant end short circuited. $S_{12}(t)$ is the current at $x=s$ under the same circumstances. Consequently from (252)

$$\begin{aligned} S_{11}(t) &= a_0(t) + 2a_{2s}(t) + 2a_{4s}(t) + \cdots \\ S_{12}(t) &= 2\{a_s(t) + a_{3s}(t) + a_{5s}(t) + \cdots\}. \end{aligned} \tag{267}$$

S_{22} is clearly equal to S_{11} by symmetry.

Now suppose that an e.m.f. $E = E(t)$ is impressed on the line at $x=0$, $t=0$, through a terminal impedance Z_1, and the distant end $(x=s)$ closed through an impedance Z_2. We suppose these terminal impedances and the actual impressed e.m.f. replaced

by the actual line voltages V_1 and V_2, impressed directly on the line at $x=0$ and at $x=s$; the terminal currents are

$$I_0(t)=\frac{d}{dt}\int_0^t S_{11}(t-\tau)V_1(\tau)d\tau \\ -\frac{d}{dt}\int_0^t S_{12}(t-\tau)V_2(\tau)d\tau, \tag{268}$$

$$I_s(t)=-\frac{d}{dt}\int_0^t S_{12}(t-\tau)V_1(\tau)d\tau \\ +\frac{d}{dt}\int_0^t S_{22}(t-\tau)V_2(\tau)d\tau. \tag{269}$$

But the current at $x=s$ is also equal to the current in the terminal impedance Z_2 in response to the terminal voltage V_2: denoting by $\alpha_2(t)$ the indicial admittance of Z_2 it is

$$I_s(t)=\frac{d}{dt}\int_0^t \alpha_2(t-\tau)V_2(\tau)d\tau. \tag{270}$$

Similarly the current entering the line at $x=0$ is the current flowing in the terminal impedance Z_1 in response to the e.m.f. $E-V^1$. Denoting by $\alpha_1(t)$ the indicial admittance of Z_1, it is

$$I_0(t)=\frac{d}{dt}\int_0^t \alpha_1(t-\tau)\{E(\tau)-V_1(\tau)\}d\tau. \tag{271}$$

Equating equations (268) and (271) and (269) and (270) we eliminate $I_0(t)$ and $I_s(t)$ and get

$$\int_0^t [S_{11}(t-\tau)+\alpha_1(t-\tau)]V_1(\tau)d\tau-\int_0^t S_{12}(t-\tau)V_2(\tau)d\tau \\ =\int_0^t \alpha_1(t-\tau)E(\tau)d\tau, \tag{272}$$

$$-\int_0^t S_{12}(t-\tau)V_1(\tau)d\tau+\int_0^t [S_{22}(t-\tau)-\alpha_2(t-\tau)]V_2(\tau)d\tau=0. \tag{273}$$

These two equations are simultaneous integral equations of the Poisson type in V_1 and V_2, which completely determine these voltages provided the admittances and the impressed voltages are known. They therefore represent the application of a new type of integral equation to the problem of electric circuit theory.

The numerical solution of the general case, either by (248) or (272–273) is necessarily laborious when the terminal impedances are complicated and is only justified when the technical importance of the problem is considerable. I wish, however, to empha-

size two points in this connection: the numerical solution is always entirely possible and, compared with other and older forms of solution, enormously simpler. One has only to inspect the classical forms of solution of problems of this type to realize the truth of this last statement.

I shall now give two applications of equations (272–273) to specific problems, in one of which an approximate solution of the integral equation can be gotten, and in the other of which numerical integration is applied.

Problem I. Given a non-inductive cable of distributed constants C and R and length s, with unit e.m.f. applied at $x=0$, while at $x=s$ the cable is closed by a condenser C_0. Required the terminal voltage $V(t)$ across the condenser C_0.

We first write down the short-circuit indicial admittances of the cable; from equation (168) of a preceding chapter and equation (267) they are:—

$$\begin{aligned} S_{11}(t)&=S_{22}(t)\\ &=\sqrt{\frac{C}{\pi Rt}}\left\{1+2e^{-\frac{4\beta}{t}}+2e^{-\frac{16\beta}{t}}+2e^{-\frac{36\beta}{t}}+\cdots\right\}, \end{aligned} \tag{274}$$

$$\begin{aligned} S_{12}(t)&=S_{21}(t)\\ &=2\sqrt{\frac{C}{\pi Rt}}\left\{e^{-\frac{\beta}{t}}+e^{-\frac{9\beta}{t}}+e^{-\frac{25\beta}{t}}+\cdots\right\}, \end{aligned} \tag{275}$$

where $\beta=s^2RC/4$.

Now the current at $x=s$ is equal to

$$S_{12}(t)-\frac{d}{dt}\int_0^t V(\tau)S_{22}(t-\tau)d\tau.$$

It is also the condenser current due to the voltage $V(t)$; that is

$$C_0\frac{d}{dt}V(t).$$

Equating the two expressions and integrating we get

$$C_0V(t)=\int_0^t S_{12}(\tau)d\tau-\int_0^t V(\tau)S_{22}(t-\tau)d\tau \tag{276}$$

which is the integral equation of the problem. In order to get an approximate solution without detailed computation we assume that the cable is long. In this case the leading terms of (274) and (275) are large compared with the terms following: Furthermore $S_{12}(t)$ builds up very slowly while $S_{22}(t)$ is a rapidly varying

function. A good approximation therefore results if we take $V(\tau)$ outside the integral sign in (276) and write

$$C_0V(t)=\int_0^t S_{12}(\tau)\,d\tau-V(t)\int_0^t S_{22}(\tau)d\tau$$

whence

$$V(t)=\frac{1}{C_0}\,\frac{\displaystyle\int_0^t S_{12}(t)dt}{1+\dfrac{1}{C_0}\displaystyle\int_0^t S_{22}(t)dt}. \tag{277}$$

This approximation is quite good for long cables and shows the way $V(t)$ builds up quite truthfully. We see that V is initially zero, and builds up ultimately to unity. For large values of t, it becomes

$$V(t)=\frac{\displaystyle\int_0^t S_{12}(t)dt}{\displaystyle\int_0^t S_{22}(t)dt}. \tag{278}$$

This is the approximate formula also for the open circuit voltage, as may be seen by setting $C_0=0$ in (277).

In electric circuit problems, it is often sufficient, as implied above, to know qualitatively the behavior of an electric system without going through the labor of detailed computation. For this purpose the formulation of the problem as a Poisson Integral Equation is particularly well adapted. A simple example will be given, which can be checked from the known solution. Suppose that we required the voltage V at point x of an infinitely long transmission line (L,R,C) in response to a unit e.m.f. impressed at $x=0$. This is, of course, known from formula (211-a): we shall here be concerned, however, with the approximate solution from the Poisson integral equation of the problem.

If $a_x(t)$ denote the indicial admittance of the line at point x, then the current at point x is simply $a_x(t)$, which is given by formula (210-a). But if $V(t)$ is the voltage at point x, the current is also given by

$$d\int_0^t V(\tau)a_0(t-\tau)d\tau.$$

Equating these two expansions, we get the integral equation of the problem

$$\frac{d}{dt}\int_0 V(\tau)a_0(t-\tau)d\tau=a_x(t).$$

Now if we write $T = \rho t - A$ where $\rho = R/2L$ and $A = \frac{xR}{2}\sqrt{\frac{C}{L}}$, then

$$a_x = \sqrt{\frac{C}{L}}e^{-(T+A)}I_0\sqrt{T(T+2A)}, \quad T>0,$$

and in terms of the relative time T, the integral equation is reducible to

$$\frac{d}{dT}\int_0^T V(T-\tau)e^{-\tau}I_0(\tau)d\tau = e^{-(T+A)}I_0(\sqrt{T(T+2A)})$$

while the exact formula for V is by (211-a)

$$V(T) = e^{-A} + Ae^{-A}\int_0^T \frac{e^{-\tau}I_1(\sqrt{\tau(\tau+2A)})}{\sqrt{\tau(\tau+2A)}}d\tau.$$

From the integral equation it is easy to establish superior and inferior limits for $V(T)$; it is

$$V(T) \leqq e^{-A}\frac{I_0(\sqrt{T(T+2A)})}{I_0(T)} = V_a(T),$$

$$\geqq \frac{\int_0^T e^{-T}I_0(T)V_a(T)dT}{\int_0^T e^{-T}I_0(T)dT} = V_b(T).$$

Both formulas give the correct initial and final values of V; namely e^{-A} and unity. Since V lies between V_a and V_b, the mean value $(V_a+V_b)/2$ also has correct initial and final values and should be a better approximation than either. The table given below shows the orders of approximation obtainable from the case where $A=3$. It is evident from this table that the foregoing approximate formulas exhibit the form of $V(T)$ qualitatively in a quite satisfactory manner.

T	V_a	V_b	$\frac{1}{2}(V_a+V_b)$	V
0	0.05	0.05	0.05	0.05
2	0.25	0.12	0.18	0.17
4	0.39	0.19	0.29	0.26
6	0.50	0.23	0.36	0.32
8	0.57	0.27	0.42	0.37
10	0.64	0.31	0.47	0.41
12	0.69	0.34	0.51	0.44
15	0.74	0.38	0.56	0.48
18	0.78	0.41	0.60	0.52

Problem II. Our second illustrative problem may be stated as follows:—A unit e.m.f. is impressed on a transmission line of length s and distributed constants L,R,C. At $x=s$ the line is closed by a resistance R_0 in parallel with an inductance L_0. Required the current in the terminal resistance. If $V(t)$ denotes the terminal voltage, the current at $x=s$ is given by

$$S_{12}(t)-\frac{d}{dt}\int_0^t V(\tau)S_{22}(t-\tau)d\tau.$$

It is also equal to the current flowing into the terminal impedance; that is

$$\frac{1}{R_0}V(t)+\frac{1}{L_0}\int_0^t V(\tau)d\tau.$$

Equating and rearranging

$$\left[\frac{1}{R_0}+S_{22}(0)\right]V(t)=S_{12}(t)-\int_0^t V(\tau)\left[\frac{1}{L_0}+S'_{22}(t-\tau)\right]d\tau. \quad (279)$$

Now the short circuit admittances S_{22} and S_{12} are given by formula (210-a) of a preceding chapter, and $S_{22}(0)=\sqrt{C/L}$. In order to apply numerical integration to (279), numerical values must be assigned to the constants. We take

$$\begin{aligned}
R_0&=\sqrt{L/C}=1935 \text{ ohms},\\
L_0&=0.4 \text{ henry},\\
\frac{R}{2L}&=\rho=292,\\
v&=1/\sqrt{LC}=1.105\times10^4,\\
s&=100.
\end{aligned}$$

The results of the numerical evaluation of equation (279), with these values inserted, is shown in Fig. 30. The voltage is identically zero until $vt=100$; $t=100/v$ is the time of propagation of the line. At that instant it jumps to the value $e^{-\rho t}=e^{-100\rho/v}$ and then begins to die away rapidly due to the draining action of the inductance. The effect of secondary reflection is insignificant and therefore not shown. The current in the terminal resistance is V/R_0 so that it is given by the same curve.

I have reserved until the last the exposition of the *expansion theorem solution* as applied to transmission lines with terminal impedances, for the reason that it is the least powerful and the most restricted, although most closely resembling the classical form of solution. Furthermore, it does not represent the

sequence of physical phenomena, in fact it is not a *wave* solution, but a solution in terms of normal or characteristic vibrations. In practical application its usefulness is restricted to the non-inductive cable.

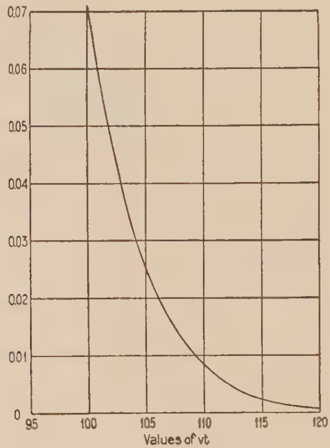

FIG. 30.—Voltage across terminal impedance on smooth line.

It will be recalled that the expansion theorem solution is formulated as follows:—

If

$$A = 1/Z(p)$$

is the operational equation of the problem, then the explicit solution is

$$A(t) = \frac{1}{Z(0)} + \sum_{1}^{n} \frac{e^{p_k t}}{p_k Z'(p_k)}$$

where $p_1, p_2 \ . \ . \ .$ are the roots of the equation $Z(p) = 0$.

Let us apply this formula to the case of a line of length s, with unit e.m.f. directly applied at $x = 0$, and line short circuited at $x = s$. Referring to equation (244) and putting $\lambda = \mu_1 = \mu_2 = 1$ we get

$$A_x = \frac{1}{K} \frac{\cosh \gamma(s-x)}{\sinh \gamma s} = \frac{1}{Z_x(p)} \tag{280}$$

as the operational formula of the problem. This can be written as

$$A_x = (Cp+G)\frac{\cosh \gamma(s-x)}{\gamma \sinh \gamma s} = \frac{1}{Z_x(p)} \tag{281}$$

where in the general case,

$$\gamma = \sqrt{(Lp+R)(Cp+G)}. \tag{282}$$

The values of γ for which $Z_x(p)$ vanishes are the roots of the transcendental equation

$$\sinh \gamma s = 0$$

excluding zero. These roots are infinite in number: Let γ_m be the m^{th} root; then

$$\gamma_m = i\frac{m\pi}{s}, \ m = 1, 2, \ \ldots . \ \infty. \tag{283}$$

The corresponding values of p_m are then gotten by solving (282) for p and writing $\gamma = \gamma_m$.

The explicit solution of the operational equation (281) is then

$$\begin{aligned} A_x(t) &= \frac{1}{Z_x(0)} + \sum \left(C + \frac{G}{p_m}\right) \frac{\cosh \gamma_m(s-x)}{s\gamma_m \dfrac{d\gamma_m}{dp_m} \cosh \gamma_m s} e^{p_m t}, \\ &= \frac{1}{Z_x(0)} + \sum (C + G/p_m) \frac{\cosh \gamma_m x}{s\gamma_m \dfrac{d\gamma_m}{dp_m}} e^{p_m t} \end{aligned} \tag{284}$$

Let us apply this to the non-inductive, non-leaky cable in which $L = G = 0$ and $\gamma = \sqrt{RCp}$, so that

$$p_m = \gamma_m^2 / RC = -\frac{m^2\pi^2}{s^2 RC}.$$

and

$$\gamma_m \frac{d\gamma_m}{dp_m} = \frac{RC}{2}.$$

Also $Z_x(0) = sR$. We thus get

$$A_x(t) = \frac{1}{sR} + \frac{2}{sR} \sum_{m=1}^{\infty} \cos\frac{m\pi}{s} x . e^{-\frac{m^2\pi^2}{s^2 RC} t}. \tag{285}$$

This is a thoroughly practical formula for computation, owing to the rapid convergence of the series. In fact, for this particular line termination chosen, it is probably the simplest and most easily computed form of solution. These advantages depend, however, strictly on two facts. First, the fact that the line is taken as non-inductive and secondly, that the terminations chosen are those of a short circuit. In fact, as we shall see, it is

only in the case of the non-inductive cable that this type of solution is of any practical value.

There is one other point which should be carefully observed in connection with this solution (285). This is that it is not expressed in terms of a series of direct and reflected waves, corresponding to the sequence of physical phenomena, but in terms of *normal* or *characteristic vibrations*. This point will be returned to later.

Let us now attempt to apply this type of solution to the transmission line, L,R,C,G. Writing

$$\rho=\frac{R}{2L}+\frac{G}{2C},$$

$$\sigma=\frac{R}{2L}-\frac{G}{2C},$$

$$v=1/\sqrt{LC}$$

we have

$$\gamma^2=\frac{1}{v^2}[(p+\rho)^2-\sigma^2]$$

whence

$$p_m=-\rho\pm v\sqrt{\gamma_m^2+\frac{\sigma^2}{v^2}}$$

$$=-\rho\pm iv\sqrt{\left(\frac{m\pi}{s}\right)^2-\frac{\sigma^2}{v^2}},\quad m=1.2,\ldots$$

$$\gamma_m\frac{d\gamma_m}{dp_m}=\frac{1}{v^2}(p_m+\rho)$$

$$=\pm\frac{i}{v}\sqrt{\left(\frac{m\pi}{s}\right)^2-\frac{\sigma^2}{v^2}}.$$

Setting $G=0$ for simplicity and substituting in (284) we get, after easy simplifications,

$$A_x(t)=\frac{1}{sR}+\frac{2vC}{s}\sum\frac{\cos\left(\frac{m\pi}{s}x\right)}{\sqrt{\left(\frac{m\pi}{s}\right)^2-\frac{\rho^2}{v^2}}}\sin\left(vt\sqrt{\left(\frac{m\pi}{s}\right)^2-\frac{\rho^2}{v^2}}\right)e^{-\rho t}.\quad(286)$$

If we write

$$\sqrt{\left(\frac{m\pi}{s}\right)^2-\frac{\rho^2}{v^2}}=\mu_m\frac{m\pi}{s}$$

(286) can be written as

$$A_x(t)=\frac{1}{sR}+\frac{vC}{s}\sum_{\mu_m}\frac{e^{-\rho t}}{\frac{m\pi}{s}}\left\{\sin\frac{m\pi}{s}(\mu_m vt-x)+\sin\frac{m\pi}{s}(\mu_m vt+x)\right\}. \tag{287}$$

This type of solution is often referred to as a *wave* solution and the component terms of the series regarded as travelling waves. As a matter of fact it is a solution in terms of normal or characteristic vibrations, each of which is to be regarded as instantaneously produced at time $t=0$. The solution in terms of true waves has been fully discussed in the preceding.

Formula (287) is practically useless for computation on account of the slow convergence of the series (the series are only conditionally convergent), and cannot be interpreted to bring out the existence of the actual direct and reflected waves and the physical character of the phenomena it formulates. In fact, as stated above, this form of solution is useful only in connection with the non-inductive cable.

In the cases considered above we have taken the simplest possible terminations—these of short circuits in which case the roots of $Z(p)$ are easily evaluated. If, however, the line is closed by arbitrary impedances, the case is quite different, and the location of the roots becomes, except for simple impedances, and then only in the case of non-inductive cable, practically impossible. While, therefore, the expansion theorem solution can be formally written down, its actual numerical evaluation is a practical impossibility, except in a few cases. For this reason it will not be considered further here.

The physically artificial character of the expansion solution, as applied to transmission lines, may be seen from the following considerations. When a wave is sent into the line, for a finite time equal to the time of the propagation of the line, it is independent of the character of the distant termination. Yet in the expansion solution every term involves and is dependent upon the impedance constituting the distant termination. Evidently, from physical considerations, the series of component vibrations making up the complete solution must therefore so combine as to annihilate the effect of the distant termination for a finite time. The solution is, therefore, mathematically correct but physically artificial.

Note on Integral Equations.

An integral equation is defined as an equation in which the unknown function occurs under a sign of integration; the process of determining the unknown function is called solving the equation.

Integral equations are of great importance in mathematical physics and in recent years very considerable work has been done on them from the standpoint of pure analysis.

The types of integral equations with which we are concerned in the present work are *Laplace's equation*

$$F(p)=\int_0^\infty e^{-pt}f(t)dt$$

and *Poisson's equation*

$$\phi(x)=f(x)+\int_0^x \phi(y)K(x-y)dy.$$

But little work has been done on Laplace's equation from the standpoint of pure analysis; its most extensive and useful applications appear to be in connection with the operational calculus. Practical methods of solution are extensively discussed in the text.

We shall now briefly discuss the solution of Poisson's equation.

The formal series solution, which is absolutely convergent, is obtained by successive substitution. Thus suppose we write

$$\phi(x)=\phi_0(x)+\phi_1(x)+\phi_2(x)+\cdots$$

and define the terms of the series in accordance with the scheme

$$\phi_0(x)=f(x),$$

$$\phi_1(x)=\int_0^x \phi_0(y)K(x-y)dy,$$

$$\phi_2(x)=\int_0^x \phi_1(y)K(x-y)dy, \text{ etc.},$$

the resulting series satisfies the integral equation and is absolutely convergent. It is, however, practically useless for computation or interpretation.

A power series solution, when it exists, can be gotten by repeated differentiation; thus

$$\phi(0)=f(0),$$

$$\phi'(x)=f'(x)+\phi(0)K(x)+\int_0^x \phi'(x-y)K(y)dy,$$

$$\phi'(0)=f'(0)+\phi(0)K(0)$$

In this way all the derivatives at $x=0$ are calculable; let them be denoted by ϕ_0, ϕ_1, ϕ_2 : . . Then

$$\phi(t)=\phi_0+\phi_1\frac{x}{1!}+\phi_2\frac{x^2}{2!}+\cdots$$

This form of solution, also, is of limited practical usefulness, except for small values of x.

A number of mathematicians, including Whittaker and Bateman, have studied the question of numerical solution and suggested other processes. After quite extensive study of the question, however, the writer is of the opinion that point-by-point numerical integration like that discussed in the text is, in general, the most practical, rapid and accurate method of numerical solution. This judgment is confirmed by G. Prasad who, in a paper on the Numerical Solution of Integral Equations before the International Mathematical Congress (Toronto, 1924), discusses the whole question and arrives at the same conclusion.

In the text, numerical integration is carried out in accordance with Simpson's Rule. It is possible, of course, to employ more complicated and refined formulas for approximate quadrature. It is the writer's opinion that this is hardly justified in practical problems and that the required accuracy is more simply obtained by employing smaller intervals.

APPENDIX TO CHAPTER IX

Note on Conventions as to Signs in Networks.

In the network shown on page 140 the arrows indicate the directions chosen as positive in the network itself, quite regardless of the presence of any e.m.fs. and currents.

The sign attributed to a current, an e.m.f., or a voltage is positive if the current, e.m.f., or voltage is in the positive direction; otherwise the sign is negative.

Stated more fully:

A current at a specific point (at a specific instant of time) is positive if it is flowing in the positive direction; negative if flowing in the negative direction.

An e.m.f. or a voltage between two points is positive if the potential increases in the positive direction between the two points; negative if the potential increases in the negative direction. (It may be noted that this convention makes the sign of a voltage the same as the sign of that e.m.f. which could be inserted between the two points without producing any effects in the network.)

CHAPTER X

INTRODUCTION TO THE THEORY OF VARIABLE CIRCUITS[1]

In the preceding chapters it has everywhere been assumed that the networks are *invariable:* that is to say, that the constants and connections of the network do not vary or change with time. In many important technical problems, however, we wish to know, not merely what happens when an electromotive force is applied to an invariable network, but the effect of suddenly changing a circuit constant or of introducing a variable circuit element. In the present chapter we shall show that this type of problem can be dealt with by a simple extension of the methods discussed in the preceding chapters.

The simplest and at the same time one of the most technically important problems of this type is the effect of sudden short circuits and sudden open circuits on an energized network or system. This type of problem will serve as an introduction to the more general theory.

The Sudden Short Circuit.—Consider the network shown in Fig. 31. This network, which for generality is supposed to consist

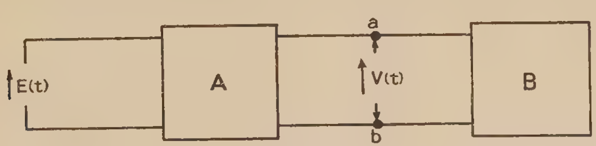

Fig. 31.

of two parts A and B, indicated schematically, is energized by an electromotive force $E(t)$ which produces a voltage $V(t)$ between the points ab. The voltage $V(t)$ is calculable by usual methods from $E(t)$ and the constants and connections of the network, supposed to be specified.

[1] The material in this chapter is largely taken from a paper by the writer on "Theory and Calculation of Variable Electrical Systems," Phys. Rev. Feb. 1921.

We now suppose that, at reference time $t=0$, a short circuit is suddenly placed across ab; and require the effect of this short circuit on the distribution of currents in the network. The solution of this problem is based on the following proposition:

The effect of the short circuit is precisely the same as the insertion at time $t=0$ of a voltage $-V(t)$, equal and opposite to $V(t)$, between points a and b.

The resultant currents in the system for $t \geqq 0$ are then composed of two components:—

(1) The currents which would exist in the invariable network, in the absence of the short circuit, due to the impressed source $E(t)$. These are calculable by usual methods.

(2) The currents due to the electromotive force $V(t)$ inserted at time $t=0$, between the points a and b. These are also calculable by usual methods, since $V(t)$ is itself known from the primary distribution of currents and charges.

By the preceding analysis we have succeeded, therefore, in reducing the problem of a sudden short circuit, to the determination of the currents in an *invariable* network in response to a suddenly impressed electromotive force: that is, the problem to which the preceding chapters have been devoted.

The Sudden Open Circuit.—The problem of a sudden open circuit in any part of a network can be dealt with in a precisely

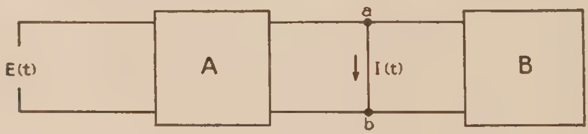

FIG. 32.

analogous manner, although the actual calculation of the resultant current and voltage distribution is mathematically more complicated. Consider the network shown in Fig. 32. Here the network is supposed to be energized by an electromotive force $E(t)$ which produces a current $I(t)$ in the *invariable* network in branch ab. We require the effect of suddenly opening this branch. The solution of this problem depends on the following proposition.

The effect of opening branch ab at reference time $t=0$ is the same as suddenly inserting at time $t=0$, a voltage $V(t)$ which produces in branch ab a current—$I(t)$ equal and opposite to the current which would exist in the branch in the absence of the open circuit.

While this proposition is precisely analogous to the corresponding proposition in the case of a sudden short circuit, it does not *explicitly* determine the voltage $V(t)$, which must be calculated as follows:

Let the driving point indicial admittance of the network, as seen from branch ab be denoted by $A_{ab}(t)$. Then, from the preceding proposition, it follows at once that $V(t)$ is given by

$$\frac{d}{dt}\int_0^t V(\tau)A_{ab}(t-\tau)d\tau=-I(t), t>0.$$

This is a Poisson integral equation in $V(t)$, from which $V(t)$ is calculable. With $V(t)$ determined, the currents in any part of the network are calculable by usual methods, and consist of two components:—

(1) The current distribution in the network due to the impressed source $E(t)$ *in the absence of the open circuit.*

(2) The current distribution due to the electromotive force $V(t)$ inserted in branch ab at time $t=0$.

As in the case of the sudden short circuit, we have thus reduced the problem of a sudden open circuit to the determination of the current distribution in an *invariable* network in response to a suddenly impressed electromotive force.

Variable Circuit Elements.—In the preceding cases of sudden open and short circuits it will be observed that the network changes discontinuously from one invariable state to another. A more general case, and one which includes the preceding as limiting cases, is presented by a network which includes a variable circuit element: that is, a circuit element which varies, continuously or discontinuously, with time. A network which includes such a variable circuit element will be called a *variable network.* Variable circuit elements of practical importance are the microphone transmitter, which consists of a variable resistance, varied by some source of energy outside the system; the condenser transmitter, which consists of a condenser of variable capacity; and the induction generator, in which the mutual inductance between primary and secondary, or stator and rotor, is varied by the motion of the latter. The case of a variable resistance will serve as an introduction to the general theory of such variable networks.

Consider a network, energized by a source $E(t)$ in branch 1, and containing a variable resistance element $r(t)$ in branch n.

The functional notation $r(t)$ indicates that the resistance r varies with time. Let $I_n(t)$ denote the current in branch n, and assume that the network is in equilibrium prior to the reference time $t=0$. The mathematical theory of this network depends on the following proposition:—

The network described above can be treated as an invariable network by eliminating the variable resistance element $r(t)$ and inserting an electromotive force $-r(t)I_n(t)$: that is, an electromotive force equal and opposite to the potential drop across the variable resistance element. Consequently the current in the variable resistance branch is determined analytically by the integral equation

$$I_n(t)=\frac{d}{dt}\int_0^t E(\tau)A_{1n}(t-\tau)d\tau-\frac{d}{dt}\int_0^t r(\tau)I_n(\tau)A_{nn}(t-\tau)d\tau. \quad (288)$$

The first component is simply the current $I_0(t)$ which would exist in the variable branch if the variable element were absent; hence, dropping the subscript n for convenience, the current in the variable branch is given by the integral equation

$$I(t)=I_0(t)-\frac{d}{dt}\int_0^t r(\tau)I(\tau)A(t-\tau)d\tau \quad (289)$$

and the voltage across the variable element by

$$v(t)=r(t)I(t). \quad (290)$$

Having determined $I(t)$ and $v(t)$ from this integral equation, the distribution of currents in the network is calculable as that due to a source $E(t)$ in branch 1 and a source $v(t)$ in branch n of the *invariable network:* that is, the network with the variable resistance element eliminated.

A very simple example will serve to illustrate the foregoing:—

Into a circuit of unit resistance, and inductance $L=1/a$, in which a steady current I_0 is flowing, a resistance r is suddenly inserted at time $t=0$: required the resultant current $I(t)$. In this case we have:

$$\begin{aligned} A(t) &= \text{indicial admittance of unvaried circuit} \\ &= 1-e^{-at}, \\ r(t) &= r, \end{aligned}$$

and the integral equation of the problem is:

$$\begin{aligned} I(t) &= I_0-r\frac{d}{dt}\int_0^t (1-e^{-ay})I(t-y)dy \\ &= I_0-ra\int_0^t I(t-y)e^{-ay}dy. \end{aligned}$$

If the solution is carried out as indicated by (291) below, and if the notation $at=x$ is introduced, we get without difficulty

$$I(t)=I_0\{1-r(1-e_1(x)e^{-x})+r^2(1-e_2(x)e^{-x})-r^3(1-e_3(x)e^{-x})+\cdots\},$$

where the function $e_n(x)$ is defined as:

$$e_n(x)=1+x/1!+x^2/2!+x^3/3!+\cdots+x^{n-1}/(n-1)!$$
$$=\text{first } n \text{ terms of the exponential series.}$$

For all finite values of the resistance increment r the series can be summed by aid of the identity

$$1-e_n(x)e^{-x}=\int_0^x dx\, e^{-x}\, x^{n-1}/(n-1)!$$

Substitution of this identity gives

$$I(t)=I_0\left(1-r\int_0^x e^{-(1+r)x}dx\right)$$
$$=I_0\frac{1+re^{-(1+r)x}}{1+r}.$$

Equation (289) is an integral equation of the Volterra type, which includes the Poisson integral equation as a special case. Its formal series solution is obtained as follows:—Assume a series solution of the form

$$I(t)=I_0(t)-I_1(t)+I_2(t)-I_3(t)+\cdots \tag{291}$$

and define the terms of the series by the scheme

$$\begin{aligned} I_1(t)&=\frac{d}{dt}\int_0^t r(\tau)I_0(\tau)A(t-\tau)d\tau \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ I_{k+1}(t)&=\frac{d}{dt}\int_0^t r(\tau)I_k(\tau)A(t-\tau)d\tau. \end{aligned} \tag{292}$$

Direct substitution shows that this series satisfies the integral equation. Furthermore, it is easily shown that it is absolutely convergent.

While this series solution is not, in general, well adapted for numerical calculations, it throws a good deal of valuable light on the ultimate character of the oscillations in the important case where $E(t)$ and $r(t)$ both vary sinusoidally with time. In this case, if the frequency of the applied e.m.f. be denoted by F and that of the resistance variation by f, it is easy to show that the

current $I_0(t)$ in the unvaried circuit is ultimately[1] a steady state current of frequency F. This follows from the fact that the definite integral which defines the current $I_0(t)$ is resolvable into the ultimate steady state current corresponding to an applied force of frequency F, and the accompanying transient oscillations which ultimately die away. The fictitious e.m.f. which may be regarded as producing the component current $I_1(t)$ is $r(t)I_0(t)$; this is ultimately the product of the two frequencies F and f, and therefore resolvable into two terms of frequency $F+f$ and $F-f$ respectively. Carrying through this analysis, it is easy to show that each component current is ultimately a steady state but poly-periodic oscillation, as indicated in the following table:

Component Current	Frequency
I_0	F,
I_1	$F+f, F-f$,
I_2	$F+2f, F, F-2f$,
I_3	$F+3f, F+f, F-f, F-3f$,
I_4	$F+4f, F+2f, F, F-2f, F-4f$.

It is of importance to observe that the component currents involve, from a mathematical standpoint, multiple integrals of successively higher orders, the nth component $I_n(t)$ involving a multiple integral of the nth order with respect to $I_0(t)$. Consequently the successive currents require longer and longer intervals of time to build up to their proximate steady state values, so that the time required for the resultant steady state to be reached cannot be inferred from the time constant of the unvaried circuit.

From the preceding table it will be seen that the ultimate steady state current is obtained by rearranging the series $I_0+I_1+I_2$ and is of the form

$$\sum_{n=-\infty}^{+\infty} A_n \cos(\Omega+n\omega)t+B_n \sin(\Omega+n\omega)t$$

where $\Omega=2\pi F$ and $\omega=2\pi f$.

[1] It hardly seems necessary to remark that the reference time $t=0$ is purely arbitrary and that the resistance variation may start at such a time thereafter that $I_0(t)$ may be regarded as steady state during the entire time interval in which we are interested. Going farther, if we confine our attention to sufficiently large values of t, the whole process may be treated as steady state.

It is interesting to note that this series comes within the definition of a Fourier series only when $F=0$ or an exact multiple of f. The steady state solution is of very considerable importance and is considered in more detail in a succeeding section.

From the foregoing we deduce an outstanding distinction between the variable and invariable networks. In the latter the currents are ultimately of the same frequency as the impressed e.m.f., whereas in the former they are ultimately of an infinite series of frequencies.

In the preceding example, the variable impedance element is a resistance $r(t)$. If the variable element is taken as an *inductance* $\lambda(t)$ the voltage, corresponding to equation (290) is

$$\frac{d}{dt}\lambda(t)I(t).$$

The case of a variable capacity element is handled as follows: Let $1/C=S$ and assume that S is variable; thus, $S=S_0+\sigma(t)$. The drop across the variable condenser element is then

$$v(t)=\sigma(t)\int_0^t I(t)dt.$$

Similarly a variable mutual inductance $\mu(t)$ between branches m and n produces the voltages

$$\frac{d}{dt}\mu(t)I_n(t)$$

in branch m, and

$$\frac{d}{dt}\mu(t)I_m(t)$$

in branch n. This case may be illustrated by:

The Induction Generator Problem

In a sufficiently general form, this problem, which includes the fundamental theory of the dynamo, may be stated as follows:

Given an invariable primary and secondary circuit with a variable mutual inductance $Mf(t)$ which is an arbitrary but specified time function, and let the primary be energized by an e.m.f. $E(t)$ impressed on the circuit at the reference time $t=0$: required the primary and secondary currents.

In operational notation the problem may be formulated by the equations:

$$Z_{11}I_1-pMf(t)I_2=E(t),$$
$$-pMf(t)I_1+Z_{22}I_2=0,$$

in which Z_{11} and Z_{22} are the self impedances of the primary and secondary respectively; $Mf(t)$ is the variable mutual inductance; $E(t)$ is the applied e.m.f. in the primary; and p denotes the differential operator d/dt. By aid of the fundamental formula these equations may be written down as the following simultaneous integral equations:

$$I_1(t)=\frac{d}{dt}\int_0^t dyA_{11}(t-y)\left(E(y)+M\frac{d}{dy}[f(y)I_2(y)]\right)$$
$$I_2(t)=M\frac{d}{dt}\int_0^t dyA_{22}(t-y)\frac{d}{dy}[f(y)I_1(y)].$$

In these equations, $A_{11}(t)$ and $A_{22}(t)$ denote the indicial admittances of the primary and secondary circuits respectively (when $M=0$): that is, the currents in these circuits in response to a unit e.m.f. (zero before, unity after time $t=0$). We assume, of course, that they are known or can be determined by usual methods.

It follows at once that the formal solution of these equations is the infinite series:

$$I_1(t)=X_0(t)+X_2(t)+X_4(t)+\cdots+X_{2n}(t)+\cdots$$
$$I_2(t)=Y_1(t)+Y_3(t)+Y_5(t)+\cdots$$

in which the successive terms of the series are defined as follows:

$$X_0(t)=\frac{d}{dt}\int_0^t dyA_{11}(t-y)E(y)=I_0(t),$$
$$Y_1(t)=M\frac{d}{dt}\int_0^t dyA_{22}(t-y)\frac{d}{dy}[f(y)X_0(y)],$$
$$X_2(t)=M\frac{d}{dt}\int_0^t dyA_{11}(t-y)\frac{d}{dy}[f(y)Y_1(y)],$$
$$Y_3(t)=M\frac{d}{dt}\int_0^t dyA_{22}(t-y)\frac{d}{dy}[f(y)X_2(y)], \text{ etc.}$$

In the light of formula

$$I(t)=\frac{d}{dt}\int_0^t f(y)A(t-y)dy$$

the physical interpretation of the series solutions follows at once: thus, $X_0(t)$ is equal to the current $I_0(t)$ flowing in the *isolated* primary in response to the applied e.m.f. $E(t)$; the first component current $Y_1(t)$ in the secondary is equal to the current which would flow in the isolated secondary in response to the applied e.m.f. $M(d/dt)f(t)X_0(t)$; $X_2(t)$, the second component current

in the primary, is equal to the current in the isolated primary in response to the applied e.m.f. $M(d/dt)f(t)Y_1(t)$; etc. The resultant currents are thus represented as built up by a to-and-fro interchange of energy between primary and secondary, or by a series of successive reactions. In the important case where the applied e.m.f. and the variation of mutual inductance are both sinusoidal time functions, of frequency F and f respectively, it is easy to show that each component current becomes ultimately equal to a set of periodic steady state currents. Thus the component X_0 is ultimately singly periodic, of frequency F; Y_1 is ultimately doubly periodic, of frequencies $F+f$ and $F-f$; X_2 triply periodic, of frequencies $F+2f$, F and $F-2f$; Y_3 quadruply periodic, of frequencies $F+3f$, $F+f$, $F-f$, $F-3f$; etc.

The Solution for the Steady State Oscillations.—For the very important case of periodic applied forces and periodic variations of circuit elements we are often concerned exclusively with the ultimate steady state of the system, and not at all with the mode in which the steady state is approached: that is, attention is restricted to the periodic oscillations which the system executes after transient disturbances have died away. In this case, if the periodic variations of circuit elements are sufficiently small, the required steady state is obtained in the form of a series by replacing each term of the complete series solution by its ultimate steady state value; a process which is very simple in view of the physical significance of each term of the latter series. The appropriate procedure will be briefly illustrated in connection with the variable resistance element. In view of the fact that we are concerned only with the ultimate steady state oscillations, we can base the solutions on the symbolic equation

$$I=I_0-\frac{r(t)}{Z}I. \tag{293}$$

Here $r(t)$ is the variable resistance element; I_0 is the current which would flow in the absence of the resistance variation; and Z is a generalized impedance of the network, as seen from the variable branch. Its precise significance and functional form is given below.

We now suppose that I_0 is given by

$$I_0=J_0e^{i\Omega t} \qquad \text{(real part)} \tag{294}$$
$$=\tfrac{1}{2}(J_0e^{i\Omega t}+\bar{J}_0e^{-i\Omega t}) \tag{295}$$

where the bar indicates the conjugate imaginary of the unbarred symbol, so that (295) is entirely real. Correspondingly the variable resistance will be taken as

$$\begin{aligned} r(t) &= \frac{r}{2}(e^{i\omega t}+e^{-i\omega t}) \\ &= re^{i\omega t} \qquad \text{(real part)} \\ &= r \cos \omega t. \end{aligned} \tag{296}$$

Here r is taken as a pure real quantity, which fixes the size of the resistance variation. No loss of generality is involved in this, since it merely involves referring the time scale to the zero of the resistance variation.

The symbolic impedance Z, as employed in the theory of alternating currents, will depend on the frequency and is, in general, a complex quantity. Its value at frequency $\Omega/2\pi$ will be denoted by

$$Z(i\Omega) = Z_0$$

while its value at frequency $(\Omega+n\omega)/2\pi$ will be written as

$$Z(i(\Omega+n\omega)) = Z_n.$$

We now assume a series solution of (293) of the form

$$I = I_0 + I_1 + I_2 + \cdots$$

where the terms of the series are defined by the symbolic equations

$$\begin{aligned} I_1 &= -\frac{r(t)}{Z} I_0 \\ &\cdots\cdots\cdot \\ I_{n+1} &= -\frac{r(t)}{Z} I_n. \end{aligned} \tag{297}$$

Substitution shows that this series formally satisfies the equation.

Starting with the first of (287) and substituting (295) and (296) we get

$$\begin{aligned} I_1 &= -\frac{r}{4Z}(e^{i\omega t}+e^{-i\omega t})(J_0 e^{i\Omega t}+J_0 e^{-i\Omega t}) \\ &= -\frac{r}{4Z}\left\{ J_0 e^{i(\Omega+\omega)t}+\bar{J}_0 e^{-i(\Omega+\omega)t}+J_0 e^{i(\Omega-\omega)t}+\bar{J}_0 e^{-i(\Omega-\omega)t}\right\}, \end{aligned} \tag{298}$$

or

$$I_1 = -\frac{r}{2} J_0 \left\{ \frac{e^{i(\Omega+\omega)t}}{Z_1} + \frac{e^{i(\Omega-\omega)t}}{Z_{-1}} \right\}. \tag{299}$$

In (299) it is to be understood that the real part is alone to be retained.

Proceeding in a similar way with the equation

$$I_2 = -\frac{r(t)}{Z} I_1$$

we get

$$I_2 = \left(\frac{r}{2}\right)^2 J_0 \left\{ \frac{e^{i(\Omega+2\omega)t}}{Z_1 Z_2} + \frac{e^{i(\Omega-2\omega)t}}{Z_{-1} Z_{-2}} + \frac{e^{i\Omega t}}{Z_0} \left(\frac{1}{Z_1} + \frac{1}{Z_{-1}} \right) \right\}. \tag{300}$$

In this way the steady state series solution is built up term by term, the component currents being poly-periodic as indicated in a previous table on p. 164.

For sufficiently small impedance variations this method of solution works very well, and leads to a rapidly convergent solution. In other cases, however, the solution so obtained may be divergent, even when the complete series solution from which it is derived is absolutely convergent. The explanation of this lies in the fact that the steady state series so obtained is the *sum of the limits* (as t approaches infinity) of the terms of the complete series solution, whereas the actual steady state is the *limit of the sum.* These are not in general equal; in particular the former may be and often is divergent when the latter is convergent.

In view of the foregoing considerations it is of importance to develop another method of investigating the steady state oscillations which avoids the difficulties in the formal series solution. The following method has suggested itself to the writer and works very well in cases where the previous form of solution fails. It should be stated at the outset, however, that the absolute convergence of the solution to be discussed, while reasonably certain in all physically possible systems, has not been established by a rigorous mathematical investigation, which appears to present very considerable difficulties.

We start with the problem just discussed and, in view of the results of the formal series solution there obtained, assume a solution of the form:

$$I = \frac{1}{2} \sum_{-N}^{N} A_m e^{i(\Omega+m\omega)t} + \bar{A}_m e^{-i(\Omega+m\omega)t} \tag{301}$$

$$= \sum_{-N}^{N} A_m e^{i(\Omega+m\omega)t} \qquad \text{(real part).} \tag{302}$$

Here the series is supposed to extend from $m=+N$ to $m=-N$. Ultimately, however, N will be put equal to infinity. As before, the bar indicates the conjugate imaginary of the unbarred symbol and (301) is therefore entirely real.

If we now substitute (301) in the symbolic equation (293) we get, by (295) and (296),

$$\frac{1}{2}\sum\left\{A_m e^{i(\Omega+m\omega)t}+\bar{A}_m e^{-i(\Omega+m\omega)t}\right\}=\tfrac{1}{2}J_0 e^{i\Omega t}+\tfrac{1}{2}\bar{J}_0 e^{-i\Omega t}$$
$$-\frac{r}{2Z}(e^{i\omega t}+e^{-i\omega t})\sum\left\{A_m e^{i(\Omega+m\omega)t}+A_m e^{-i(\Omega+m\omega)t}\right\}. \quad (303)$$

Simplifying this equation and dropping the conjugate imaginaries gives:—

$$\sum A_m e^{i(\Omega+m\omega)t}=J_0 e^{i\Omega t}-\frac{r}{Z}\sum A_m e^{i(\Omega+(m+1)\omega)t} \quad (304)$$
$$-\frac{r}{Z}\sum A_m e^{i(\Omega+(m-1)\omega)t}.$$

Finally, if we write

$$Z(i(\Omega+m\omega))=Z_m$$

and

$$r/Z_m=h_m \quad (305)$$

and equate terms of the same frequency on the two sides of the equation, we get

$$\begin{aligned} A_N&=-h_N A_{N-1} \\ A_m&=-h_m(A_{m-1}+A_{m+1}) \quad 0<|m|<N \\ A_0&=J_0-h_0(A_{-1}+A_1). \end{aligned} \quad (306)$$

It will be observed that, by (305), starting with A_N each coefficient is determined in terms of the coefficient of the next lower index. Thus:

$$\begin{aligned} A_N&=-h_N A_{N-1} \\ A_{N-1}&=-h_{N-1}(A_{N-2}+A_N) \\ &=-\frac{h_{N-1}A_{N-2}}{1-h_{N-1}h_N}. \end{aligned} \quad (307)$$

Similarly

$$A_{N-2}=-\frac{h_{N-2}A_{N-3}}{1-h_{N-2}h_{N-1}\dfrac{1}{1-h_{N-1}h_N}}.$$

Continuing this process it is easy to show that, for positive indices (m positive),

$$A_m=-h_m C_m A_{m-1} \quad (308)$$

where C_m designates the continued fraction

$$\cfrac{1}{1-h_m h_{m+1} \cfrac{1}{1-h_{m+1} h_{m+2} \cdot \cdot \cdot \cdot \cfrac{1}{1-h_{N-1} h_N}}}$$

The procedure for the coefficient A_{-m} is precisely similar. For convenience we write $A_{-m}=A'_m$, $Z_{-m}=Z'_m$, and $r/Z_{-m}=h'_m$. In this notation we get by precisely similar procedure

$$A'_m = -h'_m C'_m A'_{m-1} \tag{309}$$

where C'_m designates the continued fraction

$$\cfrac{1}{1-h'_m h'_{m+1} \cfrac{1}{1-h'_{m+1} h'_{m+2} \cdot \cdot \cdot \cdot \cfrac{1}{1-h'_{N-1} h'_N}}}$$

We now put the index N equal to infinity and the continued fractions C_m and C'_m become infinite instead of terminating fractions.

Collecting formulas we now have

$$\begin{aligned} A_m &= -h_m C_m A_{m-1} \\ A'_m &= -h'_m C'_m A'_{m-1} \end{aligned} \tag{310}$$

and

$$A_0 = J_0 - (h_0 A_1 + h_0' A'_1)$$

whence

$$A_0 = \frac{J_0}{1-h_0 h_1 C_1 - h_0' h_1' C_1'}.$$

The coefficients are thus all determined in terms of J_0.

The practical value of this method of solution will depend, of course, on the rate of convergence of the continued fractions. While no rigorous proof has been obtained, it is believed that they are absolutely convergent for all physically possible systems, but this question certainly requires fuller investigation. Nevertheless any doubt regarding the convergence of the solution need not prevent the use of the method in a great many problems where physical considerations furnish a safe guide. For example

this method of solution, when applied to the problem of the induction generator, discussed above, leads to the usual simplified engineering theory of the induction generator and motor, besides exhibiting effects which the usual treatment either ignores or fails to recognize.

Non-linear Circuits.—In the previous examples discussed, the variations of the variable circuit elements are assumed to be specified time functions, which is the same thing as postulating that these variations are controlled by ignored forces which do not explicitly appear in the statement and equations of the problem. We distinguish another type of variable circuit element, where the variation is not an explicit time function but rather a function of the current (and its derivatives) which is flowing through the circuit. For example, the inductance of an iron-core coil varies with the current strength as a consequence of magnetic saturation. The equation of a circuit which contains such a variable element (provided it is a single valued function) may be written down in operational notation

$$ZI+\phi(I)=E(t),$$

or

$$ZI=E(t)-\phi[I(t)]. \tag{311}$$

In this equation Z is, of course, to be taken as the impedance of the invariable part of the circuit, the indicial admittance of which is denoted by the usual symbol $A(t)$.

Equation (311) may be interpreted as the equation of the current $I(t)$ in a circuit of invariable impedance Z when subjected to an applied e.m.f. $E(t)-\phi[I(t)]$; consequently, by aid of our fundamental formula, $I(t)$ is given by

$$I(t)=\frac{d}{dt}\int_0^t A(t-y)E(y)dy-\frac{d}{dt}\int_0^t A(t-y)\phi[I(y)]dy.$$

The first integral is simply the current in the invariable circuit of impedance Z in response to the applied e.m.f. $E(t)$; denoting this by $I(t)$, we have

$$I(t)=I_0(t)-\frac{d}{dt}\int_0^t A(t-y)\phi[I(y)]dy.$$

This is a *functional integral equation*, the solution of which is gotten by some process of successive approximations. For

example, provided the sequence converges, $I(t)$ is the limit as n approaches infinity of the *sequence*

$$I_0(t),\ I_1(t),\ I_2(t), \ldots,\ I_n(t),$$

where the successive terms of the sequence are defined by the relations:

$$I_1(t) = I_0(t) - \frac{d}{dt}\int_0^t A(t-y)\phi[I_0(y)]dy.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$I_{n+1}(t) = I_0(t) - \frac{d}{dt}\int_0^t A(t-y)\phi[I_n(y)]dy.$$

We shall not pursue the discussion of non-linear circuits further, in view of their mathematical complexity and their relatively specialized technical interest. The reader who is interested may, however, consult the writer's paper on Variable Electrical Systems,[1] for a fuller treatment of the subject.

[1] Phys. Rev. Feb., 1921.

CHAPTER XI

THE APPLICATION OF THE FOURIER INTEGRAL TO ELECTRIC CIRCUIT THEORY

The application of Fourier's series in electrotechnics is a commonplace; the use of the Fourier integral, however, has largely remained in the hands of professional mathematicians. An outstanding distinction between the series and the integral, from which the greater power of the latter may be inferred, is that the series represents only a periodic regularly recurrent function, whereas the integral is capable of representing a non-periodic function: in fact all types of functions, subject to certain mathematical restrictions which are usually satisfied in physical problems.

Before taking up the application of the Fourier Integral to Electric Circuit Theory, we shall very briefly review the elementary mathematics of the series and integral; for a fuller treatment the reader is referred to Byerly, Fourier's Series and Spherical Harmonics.[1]

Consider a function $\phi(t)$, which in the region $0<t<T$ is finite, single-valued and has only a finite number of discontinuities or of maxima or minima. In this region it can then be expressed as the Fourier series

$$\phi(t)=\tfrac{1}{2}A_0+\sum_{1}^{\infty}\left\{A_n\cos\left(\frac{2\pi n}{T}t\right)+B_n\sin\left(\frac{2\pi n}{T}t\right)\right\} \qquad (312)$$

where

$$\begin{aligned} A_n&=\frac{2}{T}\int_0^T\phi(t)\cdot\cos\left(\frac{2\pi n}{T}t\right)dt,\\ B_n&=\frac{2}{T}\int_0^T\phi(t)\cdot\sin\left(\frac{2\pi n}{T}t\right)dt. \end{aligned} \qquad (313)$$

[1] In this chapter the Fourier integral is approached from the view-point of its physical application and no completeness or rigour is claimed for the treatment. The mathematical theory of the Fourier integral is, of course, completely developed in treatises on the subject. The object of this chapter is merely to outline some of its applications.

An equivalent series is

$$\phi(t)=\tfrac{1}{2}F_0+\sum_{1}^{\infty}F_n\cos\left(\frac{2\pi n}{T}t-\Theta_n\right) \tag{314}$$

where

$$F_n=\sqrt{A_n^2+B_n^2}, \tag{315}$$
$$\Theta_n=\tan^{-1}(B_n/A_n).$$

This expansion is valid in the region $0<t<T$, irrespective of the form of the function elsewhere. Let us, however, assume that the function repeats itself in the period T: that is

$$\phi(t\pm kT)=\phi(t),\quad k=1,\,2,\,3\ldots N.$$

Then the expansion represents the function in the region

$$-NT<t<(N+1)T.$$

Finally if N is made infinite, the function is truly periodic and the Fourier series represents it for all positive and negative values of time.

It follows from the foregoing that, if the Fourier series represents the function for all positive and negative values of time, the function must be periodic for all positive and negative values of time; otherwise the expansion is valid only over a restricted range of time.

Now let us suppose that $\phi(t)$ is non-periodic. For convenience, in connection with subsequent applications we shall suppose that it is zero for all finite *negative* values of time, that it converges to zero as $t\rightarrow\infty$, and that

$$\int_0^\infty|\phi(t)|dt$$

exists. Such a function obviously cannot be represented by the usual Fourier series for all finite positive and negative values of time; it can be represented, however, by the limiting form assumed by the series as the fundamental period T is made infinite. That is, we can assume that the function is periodic in an infinite fundamental period and this will not affect the expansion for finite positive and negative values of time. Proceeding in this way and putting the fundamental period T equal to infinity in the limit, the Fourier series (314) becomes an infinite integral and we get

$$\phi(t)=\frac{1}{\pi}\int_0^\infty F(\omega)\cdot\cos\left(\omega t-\theta(\omega)\right)d\omega \tag{316}$$

where

$$F(\omega)=\left\{\left[\int_0^\infty \phi(t)\cos\omega t\,dt\right]^2+\left[\int_0^\infty \phi(t)\sin\omega t\,dt\right]^2\right\}^{\frac{1}{2}} \quad (317)$$

and

$$\tan\theta(\omega) = \int_0^\infty \phi(t)\sin\omega t\,dt\div\int_0^\infty \phi(t)\cos\omega t\,dt. \quad (318)$$

This is the *Fourier integral* identity of the function $\phi(t)$ and is valid for all finite positive and negative values of time.

In physical applications, particularly those to electric circuit theory, it is often convenient to employ exponential instead of trigonometric functions. The required transformation follows easily from the relation

$$e^{i\theta}=\cos\theta+i\sin\theta, \qquad i=\sqrt{-1}.$$

Thus if we write $2\pi/T=\omega_0$ the Fourier series (312) is easily reduced to the form

$$\phi(t)=\sum_{-\infty}^{+\infty}F(in\omega_0)e^{in\omega_0 t} \quad (319)$$

where

$$F(in\omega_0)=\frac{1}{T}\int_0^T \phi(\tau)e^{-in\omega_0\tau}\,d\tau. \quad (320)$$

In precisely similar manner the Fourier integral (316) can be written as

$$\phi(t)=\int_{-\infty}^{\infty} F(i\omega)\cdot e^{i\omega t}\,d\omega \quad (321)$$

$$=\frac{1}{2\pi}\int_0^\infty \phi(\tau)d\tau\int_{-\infty}^{\infty} e^{i\omega(t-\tau)}\,d\omega. \quad (322)$$

Applications to Electric Circuit Theory.—Let us assume that at time $t=-NT$, an electromotive force $E(t)$, periodic in fundamental period T, is impressed on a circuit of complex impedance $Z(i\omega)$, where ω denotes 2π times the frequency. Required the resultant current I.

For values of $t>-NT$ the electromotive force (see formula (319)) can be expressed as the Fourier series

$$E(t)=\sum_{-\infty}^{\infty}F(in\omega_0)e^{in\omega_0 t}$$

where

$$F(in\omega_0)=\frac{1}{T}\int_0^T E(\tau)e^{-in\omega_0\tau}d\tau.$$

The resultant current for $t>-NT$ is therefore

$$I=\sum_{-\infty}^{\infty}\frac{F(in\omega_0)}{Z(in\omega_0)}e^{in\omega_0 t}+\left\{\begin{array}{l}\text{transient oscillations}\\ \text{initiated at time}\\ t=-NT.\end{array}\right\}.$$

If we are concerned with the current for values of $t>0$, and if NT is made sufficiently large, the initial transients will have died away and the *complete current* for $t>0$, will be given by

$$I=\sum_{-\infty}^{\infty}\frac{F(in\omega_0)}{Z(in\omega_0)}e^{in\omega_0 t}. \tag{323}$$

This formula implies the periodic character of $E(t)$ for sufficiently large negative values of time. If, however, $E(t)$ is zero for negative values of time, we can employ the Fourier integrals (321) and (322) in precisely the same way and get, as the *complete* expression or the current for positive or negative values of time:—

$$I=\int_{-\infty}^{\infty}\frac{F(i\omega)}{Z(i\omega)}e^{i\omega t}\,d\omega \tag{324}$$

$$=\frac{1}{2\pi}\int_0^{\infty}E(\tau)d\tau\int_{-\infty}^{\infty}\frac{e^{i\omega(t-\tau)}}{Z(i\omega)}d\omega \tag{325}$$

The infinite integrals (324) *and* (325) *formulate the current in the network, specified by the impedance function* $Z(i\omega)$, *in response to an electromotive force* $E(t)$ *impressed at time* $t=0$; *they therefore mathematically formulate, by aid of the Fourier integral identity, the fundamental problem dealt with in the preceding chapters and solved by aid of the operational calculus.*

No attempt will be made here to discuss the solution of the infinite integral (325), which is usually a problem presenting formidable difficulties, even to the professional mathematician. The general method of solution is by contour integration in the complex plane and the calculus of residues. By this process it has been successfully applied to the solution of special problems, and also to deriving some general forms of solution such as the expan-

sion theorem solution.[1] Compared, however, with the operational calculus, it has no advantages from the standpoint of rigour, and lacks entirely the remarkable simplicity and directness of the Heaviside method.

In the direct solution of circuit problems, therefore, it is believed that the application of the Fourier integral is attended by few if any advantages, and presents formidable mathematical difficulties. On the other hand, there are certain types of problems encountered in circuit theory, where the Fourier integral is a powerful tool. These will be briefly discussed.

The Indicial Admittance Expressed as a Fourier Integral.—Without recourse to the general formulas (324) and (325), it is a very simple matter to formulate the indicial admittance $A(t)$ (and the Heaviside Function $h(t)$) as a Fourier Integral. To this we now proceed, starting with the known identity

$$\begin{aligned} \frac{1}{2}\left(1+\frac{2}{\pi}\int_0^\infty \frac{\sin \omega t}{\omega} d\omega\right) &= 1 \qquad t>0 \\ &= 0 \qquad t>0 \end{aligned} \tag{326}$$

Evidently this expression represents the unit function, zero before, unity after, time $t=0$.

Now let us suppose that the unit e.m.f., as formulated by (326), is impressed on a network of impedance $Z(i\omega)$, and let us write

$$\frac{1}{Z(i\omega)} = \frac{e^{i\theta(\omega)}}{Z(i\omega)} = \alpha(\omega) + i\beta(\omega) \tag{327}$$

so that $\alpha(\omega)$ and $\beta(\omega)$ are the two components of the steady state admittance at frequency $\omega/2\pi$.

Now the first term, $\frac{1}{2}$, of (326) is of zero frequency and produces a current $\frac{1}{2}\,\alpha(0)$. The integral term contains components of all frequencies, that lying between ω and $\omega + d\omega$, being

$$\frac{2}{\pi}\frac{1}{\omega} \sin \omega t . d\omega$$

and the corresponding current is

$$\frac{2}{\pi}\frac{1}{\omega|Z(i\omega)|} \sin\,(\omega t + \theta(\omega) d\omega.$$

[1] Bush, "Summary of Wagner's Proof of Heaviside's Formula." Proc. Inst. of Radio Engineers. Oct., 1917. Fry. "The Solution of Circuit Problems." (Phys. Rev. Aug., 1919.)

Consequently the total current, which is the indicial admittance, is given by

$$A(t)=\frac{1}{2}\alpha(0)+\frac{1}{2\pi}\int_0^\infty \frac{\sin(\omega t+\theta(\omega))}{\omega|Z(i\omega)|}d\omega \tag{328}$$

$$=\frac{1}{2}\alpha(0)+\frac{1}{\pi}\int_0^\infty \frac{\alpha(\omega)}{\omega}\sin\omega t\,d\omega \tag{329}$$
$$+\frac{1}{\pi}\int_0^\infty \frac{\beta(\omega)}{\omega}\cos\omega t\,d\omega.$$

The same process applies, obviously, to the Heaviside function $h(t)$, representing current, charge, potential drop, or any of the circuit variables with which we may be concerned.

Thus if the generalized steady state impedance is represented by $H(i\omega)$, and if we write

$$\frac{1}{H(i\omega)}=a(\omega)+ib(\omega),$$

we have

$$h(t)=\frac{1}{2}a(0)+\frac{1}{\pi}\int_0^\infty \frac{a(\omega)}{\omega}\sin\omega t\,d\omega \tag{330}$$
$$+\frac{1}{\pi}\int_0^\infty \frac{b(\omega)}{\omega}\cos\omega t\,d\omega.$$

Formula (329) for the indicial admittance is then simply a special case of formula (330) for the Heaviside function $h(t)$; that is, the response of the network to a unit e.m.f. impressed at time $t=0$.

By (330) we have, changing the sign of t

$$h(-t)=\frac{1}{2}a(0)-\frac{1}{\pi}\int_0^\infty \frac{a(\omega)}{\omega}\sin\omega t\,d\omega \tag{331}$$
$$+\frac{1}{\pi}\int_0^\infty \frac{b(\omega)}{\omega}\cos\omega t\,d\omega$$

By subtraction and addition of (330) and (331) we get at once

$$h(t)-h(-t)=\frac{2}{\pi}\int_0^\infty \frac{a(\omega)}{\omega}\sin\omega t \tag{332}$$

$$h(t)+h(-t)=a(0)+\frac{2}{\pi}\int_0^\infty \frac{b(\omega)}{\omega}\cos\omega t. \tag{333}$$

Now let us suppose that $t>0$; then since $h(t)$ is identically zero for negative values of t, it follows that $h(-t)$ is zero in equations (332) and (333), and we have

$$h(t)=\frac{2}{\pi}\int_0^{\infty}\frac{a(\omega)}{\omega}\sin\omega t \tag{334}$$

$$=a(0)+\frac{2}{\pi}\int_0^{\infty}\frac{b(\omega)}{\omega}\cos\omega t \tag{335}$$

provided $t>0$. It is therefore only necessary to evaluate one of the infinite integrals of equation (330). As an interesting corollary, *the behavior of the network under all circumstances is completely determined if either the real or imaginary component of the complex steady state admittance is specified over the entire frequeny range.*

The analytical evaluation of $h(t)$ from (334) or (335) is usually quite complicated, unless the infinite integral can be recognized, and involves advanced mathematical methods. Its numerical evaluation can sometimes be effected without prohibitive labor by the following process, applied to either (334) or (335). Selecting (334), it can be written as

$$h(t)=\frac{2}{\pi}\int_0^{\infty}\frac{1}{\lambda}a\left(\frac{\lambda}{t}\right)\sin\lambda.d\lambda \tag{336}$$

and when t becomes very large, this approaches

$$a(0).\frac{2}{\pi}\int_0^{\infty}\frac{1}{\lambda}\sin\lambda d\lambda=a(0) \tag{337}$$

which is, of course, the ultimate steady value of $h(t)$. Now write (336) in the form

$$\begin{aligned}h(t)=&\frac{2}{\pi}\int_0^{x}\frac{1}{\lambda}a\left(\frac{\lambda}{t}\right)\sin\lambda.d\lambda\\&+\frac{2}{\pi}\int_x^{\infty}\frac{1}{\lambda}a\left(\frac{\lambda}{t}\right)\sin\lambda.d\lambda\end{aligned} \tag{338}$$

where x may have any finite real value we please. Now having assigned some particular value to x, the first integral of (338) can be evaluated mechanically or numerically. The second integral can then be expanded asymptotically by repeated partial integrations as follows:

$$\int_0^{\infty}\frac{1}{\lambda}a\left(\frac{\lambda}{t}\right)\sin\lambda.d\lambda=-\int_x^{\infty}\frac{1}{\lambda}a\left(\frac{\lambda}{t}\right).d\cos\lambda$$

$$=-\frac{1}{x}a\left(\frac{x}{t}\right)\cos x+\int_x^{\infty}\cos\lambda.\frac{d}{d\lambda}\frac{1}{\lambda}a\left(\frac{\lambda}{t}\right).d\lambda$$

Repeating this process we get for the second integral term of (338) the series

$$\begin{aligned}\frac{2}{\pi}\cos x\left\{1-\frac{d^2}{dx^2}+\frac{d^4}{dx^4}-\cdots\right\}\frac{1}{x}a\left(\frac{x}{t}\right)\\ -\frac{2}{\pi}\sin x\left\{\frac{d}{dx}-\frac{d^3}{dx^3}\cdots\right\}\frac{1}{x}a\left(\frac{x}{t}\right)\end{aligned} \tag{339}$$

which may be used for computation for sufficiently large values of t and x.

This process for evaluating (338) is usually laborious but is often quite practicable. It will be observed that the computation becomes easier for large values of t. On the other hand, the evaluation of $h(t)$ by the operational calculus is usually simpler for small values of t. This suggests that in some problems the numerical solution may often be advantageously effected by a combination of the operational calculus and the Fourier integral, using the former to calculate $h(t)$ for small values of t, and formula (338) to calculate it for large values of t. In practical problems, therefore, the Fourier integral may be of great value in supplementing the operational calculus.

The Building-up of Alternating Currents.—The preceding analysis can be generalized and extended to the case of sinusoidal impressed forces by employing the expressions

$$\cos \omega t\left[\frac{1}{2}+\frac{1}{\pi}\int_0^\infty\frac{\sin \lambda t}{\lambda}d\lambda\right] \tag{340}$$

and

$$\sin \omega t\left[\frac{1}{2}+\frac{1}{\pi}\int_0^\infty\frac{\sin \lambda t}{\lambda}d\lambda\right]. \tag{341}$$

Formula (340) represents a function which is zero for negative values of time t, and equal to $\cos \omega t$ for positive values of time t, while (341) represents a function which is zero for $t<0$ and $\sin \omega t$ for $t>0$. *They can therefore be applied to formulating and investigating the behavior of networks in response to suddenly applied sinusoidal electromotive forces.*

The application suggested by the preceding is of considerable practical importance because in long transmission systems the calculaton of the building-up of alternating currents is extremely complicated and laborious. On the other hand an investigation of the mode of building-up is necessary in long communication systems since on the character and duration of the building-up

process, the excellence of a signal transmission system largely depends.

To formulate the problem of the building-up of alternating currents, let us suppose that an e.m.f. $E \cos \omega t$ is suddenly applied at reference time $t = 0$, to a net work or transmission system of complex steady state impedance

$$Z(i\omega) = Z|(i\omega)|e^{-iB(\omega)}.$$

It is then always possible to write the resultant current in the form,

$$I(t) = \frac{1}{2}\frac{E}{|Z(i\omega)|}\{(1+\rho)\cos(\omega t - B) + \sigma \sin(\omega t - B)\} \quad (342)$$

$$= \frac{1}{2}\sqrt{(1+\rho)^2+\sigma^2}\,\frac{E}{Z(i\omega)|}\cos(\omega t - B(\omega) - \theta) \quad (343)$$

where $\theta = \tan^{-1}\frac{\sigma}{1+\rho}$

In these formulas ρ, σ and θ are time functions which it is our problem to evaluate. Evidently ρ and σ must be -1 and 0 respectively for negative values of time t, and approach the limit 1 and 0 respectively as $t \to \infty$.

The function $\frac{1}{2}\sqrt{(1+\rho)^2+\sigma^2}$ of equation (343) is proportional to the *envelope* of the oscillations. In an engineering study of the building-up process we are often not concerned with the instantaneous values of the current but merely with their envelope. In such cases an investigation of this function is sufficient.

The mathematical formulation of ρ and σ as Fourier integrals is based on formula (340). Starting with this formula as representing a suddenly applied sinusoidal e.m.f., $\cos \omega t$, of frequency $\omega/2\pi$, and proceeding by reasoning strictly analogous to that employed in the preceding section, it can be shown that ρ and σ are given by the formulas

$$\rho = \frac{1}{\pi}\int_0^\infty \{P_\omega(\lambda) + P_\omega(-\lambda)\}\sin t\lambda\,\frac{d\lambda}{\lambda} \quad (344)$$

$$-\frac{1}{\pi}\int_0^\infty \{Q_\omega(\lambda) - Q_\omega(-\lambda)\}\cos t\lambda\,\frac{d\lambda}{\lambda}$$

$$\sigma = \frac{1}{\pi}\int_0^\infty \{Q_\omega(\lambda) + Q_\omega(-\lambda)\}\sin t\lambda\,\frac{d\lambda}{\lambda} \quad (345)$$

$$+\frac{1}{\pi}\int_0^\infty \{P_\omega(\lambda) - P_\omega(-\lambda)\}\cos t\lambda\,\frac{d\lambda}{\lambda}$$

where

$$P_\omega(\lambda)=\frac{|Z(i\omega)|}{|Z(i\omega+i\lambda)|}\cos\,[(B\omega+\lambda)-B(\omega)] \tag{346}$$

$$Q_\omega(\lambda)=\frac{|Z(i\omega)|}{|Z(i\omega+i\lambda)|}\sin\,[B(\omega+\lambda)-B(\omega)]. \tag{347}$$

By virtue of the fact that ρ and σ are identically -1 and 0 respectively for negative values of time t, it follows at once (See derivation of (332) and (333) from (330)), that we have also for *positive values of time* $(t>0)$.

$$1+\rho=\frac{2}{\pi}\int_0^\infty\{P_\omega(\lambda)+P_\omega(-\lambda)\}\sin t\lambda\,\frac{d\lambda}{\lambda} \tag{348-a}$$

$$1-\rho=\frac{2}{\pi}\int_0^\infty\{Q_\omega(\lambda)-Q_\omega(-\lambda)\}\cos t\lambda\,\frac{d\lambda}{\lambda} \tag{348-b}$$

and

$$\sigma=\frac{2}{\pi}\int_0^\infty\{Q_\omega(\lambda)+Q_\omega(-\lambda)\}\sin t\lambda\,\frac{d\lambda}{\lambda}\cdot \tag{349-a}$$

$$=\frac{2}{\pi}\int_0^\infty\{P_\omega(\lambda)-P_\omega(-\lambda)\}\cos t\lambda\,\frac{d\lambda}{\lambda}\cdot \tag{349-b}$$

For important types of transmission systems, particularly the periodically loaded line, these formulas have been successfully dealt with and solutions of a satisfactory approximate character obtained. For further details the reader is referred to a paper on "The Building-up of Sinusoidal Currents in Long Periodically Loaded Lines" (Bell System Technical Journal, Oct., 1924).

The Distortion of Signals in Communication Circuits.—In electrical communication, the transmission system must be so designed that the received signal must be a more or less faithful copy of the transmitted signal; the departure from this requirement is termed *distortion.* The Fourier integral provides a simple method of formulating the conditions for distortionless transmission, and the factors on which distortion depend.

Let $f(t)$ be the voltage impressed on the sending end of a transmission system of transfer impedance

$$Z(i\omega)\;=\;Z|(i\omega)|e^{iB(\omega)},$$

and let us suppose that $f(t)$ is expressed as the Fourier integral

$$f(t)=\frac{1}{\pi}\int_0^\infty F(\omega)\cdot\cos\,(\omega t-\theta(\omega)) \tag{350}$$

where $\omega = 2\pi$ times the frequency. The received current is then given by

$$I(t) = \frac{1}{\pi}\int_0^\infty \frac{F(\omega)}{|Z(i\omega)|} \cos\left(\omega t - \theta(\omega) - B(\omega)\right). \tag{351}$$

Now $f(t)$ of formula (350) contains components of all frequencies from zero to infinity; in other words $F(\omega)$ exists for all values of ω. Practically, however, $F(\omega)$ is negligibly small except in a range depending on the character of the signal. For example in slow speed telegraphy, the essential frequencies lie in a band between zero and 10 to 20 cycles per second. In telephony the essential frequencies occupy a band extending from a low frequency, say 50 c.p.s., to an upper limiting frequency of approximately 4000 c.p.s.

Let us suppose, therefore, that the essential frequency range of $f(t)$ lies between $\omega_1/2\pi$ and $\omega_2/2\pi$. Then equations (350) and (351) can be replaced by

$$f(t) = \frac{1}{\pi}\int_{\omega_1}^{\omega_2} F(\omega) \cos\left(\omega t - \theta(\omega)\right) \tag{352}$$

$$I(t) = \frac{1}{\pi}\int_{\omega_1}^{\omega_2} \frac{F(\omega)}{|Z(i\omega)|} \cos\left(\omega t - \theta(\omega) - B(\omega)\right). \tag{353}$$

Now let us suppose that the transmission system is so designed that over the range $\omega_1 < \omega < \omega_2$

$$|Z(i\omega)| = K$$
$$B(\omega) = \omega\tau.$$

where K and τ are constants. Formula (353) then becomes

$$I(t) = \frac{1}{\pi K}\int_{\omega_1}^{\omega_2} F(\omega) \cos\left(\omega(t-\tau) - \theta(\omega)\right). \tag{354}$$

Comparison of (352) and (354) shows at once that

$$I(t) = \frac{1}{K} f(t-\tau). \tag{355}$$

That is, the received current has precisely the same wave form as the impressed e.m.f.; it is simply changed in amplitude by the factor $1/K$, and delayed in time by the "transmission time" τ. The received signal is therefore a faithful copy of the transmitted signal and the transmission is distortionless. We thus arrive at the following important proposition.

The necessary and sufficient conditions for the distortionless transmission of signals in communication systems is that, over the essential range of frequencies contained in the transmitted signal, the transfer impedance of the transmission circuit be equalized both as regards amplitude and phase; that is, the amplitude must be constant and the phase angle linear in the frequency.

The case where the amplitude alone is equalized over the frequency range corresponding to the spectrum of the signal is covered by the following proposition.

The necessary and sufficient condition that the transmitted and received signals have the same relative energy content with respect to frequency is that the transmission system be equalized as regards amplitude over the range of frequencies contained in the signal.

It is to be observed that this condition alone, however, does not insure preservation of *wave form.* In types of signalling systems, therefore, where preservation of wave form is essential, it is necessary to equalize both phase and amplitude.

The Energy Absorbed from Transient Applied Forces.—In many technical problems, the complete solution for the instantaneous current due to suddenly applied electromotive forces, although formally straight-forward, involves a prohibitive amount of labor. In yet others, the applied forces may be random and specified only by their mean square values. In such problems a great deal of useful information is furnished by the mean power and mean square current absorbed by the network, and to the calculation of these quantities, the Fourier integral is ideally adapted. Its application depends on the following proposition, due to Rayleigh (Phil. Mag., Vol. 27, 1889, p. 466), and its corollary.

Let a function $\phi(t)$, supposed to exist only in the epoch $0 < t < T$, be formulated as the Fourier integral

$$\phi(t) = \frac{1}{\pi}\int_0^\infty |f(\omega)| \,.\, \cos\,[\omega t - \theta(\omega)]d\omega$$

where

$$f(\omega) = \left\{\left[\int_0^T \phi(t)\ \cos\ \omega t\ dt\right]^2 + \left[\int_0^T \phi(t)\ \sin\ \omega t\ dt\right]^2\right\}^{\frac{1}{2}}$$

$$\tan\ \theta(\omega) = \int_0^T \phi(t) \sin\ \omega t\ dt \div \int_0^T \phi(t)\ \cos\ \omega t\ dt.$$

Then

$$\int_0^T [\phi(t)]^2 dt = \frac{1}{\pi}\int_0^\infty |f(\omega)|^2 d\omega,$$

whereby the time integral is transformed into an integral with respect to frequency.

A corollary of this theorem is as follows:

If two functions $\phi_1(t)$, $\phi_2(t)$ supposed to exist only in the epoch $0<t<T$, are formulated by the Fourier integrals

$$\phi_1(t)=\frac{1}{\pi}\int_0^\infty |f_1(\omega)|\cdot\cos\,[\omega t-\theta_1(\omega)]d\omega,$$

$$\phi_2(t)=\frac{1}{\pi}\int_0^\infty |f_2(\omega)|\cdot\cos[\omega t-\theta_2(\omega)]d\omega,$$

then

$$\int_0^T \phi_1(t)\phi_2(t)dt=\frac{1}{\pi}\int_0^\infty |f_1(\omega)|\cdot|f_2(\omega)|\cdot\cos\,(\theta_1-\theta_2)d\omega.$$

The application of these theorems to circuit theory proceeds as follows:—

If an electromotive force $E(t)$, supposed to exist only in the epoch $0<t<T$, is applied to a network of complex impedance $Z(i\omega)=|Z(i\omega)|e^{i\beta(\omega)}$, we know from the preceding discussion of the Fourier integral, that the electromotive force $E(t)$ and current $I(t)$ are expressible as the Fourier integrals

$$E(t)=\frac{1}{\pi}\int_0^\infty |f(\omega)|\cdot\cos\,(\omega t-\theta(\omega))d\omega, \tag{356}$$

$$I(t)=\frac{1}{\pi}\int_0^\infty \frac{|f(\omega)|}{|Z(i\omega)|}\cos\,(\omega t-\theta(\omega)-\beta(\omega))d\omega.$$

It follows at once from Rayleigh's theorem that

$$\int_0^\infty I^2dt=\frac{1}{\pi}\int_0^\infty \frac{|f(\omega)|^2}{|Z(i\omega)|^2}d\omega. \tag{357}$$

Now let I_n be the current absorbed in branch n; let $z(i\omega)=|z(i\omega)|e^{i\alpha(\omega)}$ be the impedance of that branch and let $E_n(t)$ be the potential drop across that branch. It follows at once from the corollary to Rayleigh's theorem that

$$W=\int_0^\infty E_n(t)I_n(t)dt=\frac{1}{\pi}\int_0^\infty \frac{|f(\omega)|^2}{|Z(i\omega)|^2}|z(i\omega)|\cos\alpha(\omega)d\omega. \tag{358}$$

Formulas (357) and (358) formulate the mean square current and mean power absorbed by the branch of the network under consideration, and enable us to calculate these quantities, even in the case of complicated networks, with a minimum of labor. Formula (357) is particularly well adapted to computation

because the integrand is everywhere positive, permitting, in most problems, of easy numerical integration, whereas the analytical solution may be complicated.

Selective Circuits and Interference.—An important application of the preceding formulas is to the theory of selective circuits with particular reference to the protection they afford against irregular and transient disturbances such as "noise" in telephone systems and "static interference" in radio systems. In these cases the disturbance or interference is essentially random in character; its instantaneous wave form is unknown, and it can be specified only by its "frequency distribution." Under these circumstances the problem is a statistical one and the only information deducible is the mean power and mean square current absorbed by the circuit from the disturbance. This is calculable from the preceding formulas when the statistical or average frequency distribution of the disturbance is determined or specified. A brief sketch of the analytical theory follows.

Let a disturbance $f(t)$, for convenience supposed to exist only in the epoch $0<t<T$, be impressed on a network of impedance $Z(i\omega)$. If the disturbance is specified for all values of t in this epoch, it is theoretically possible to calculate $F(\omega)$ and $\theta(\omega)$ and to express $f(t)$ as the Fourier integral

$$f(t)=\frac{1}{\pi}\int_0^\infty F(\omega)\,.\cos\,(\omega t-\theta(\omega))d\omega. \qquad (359)$$

In any case even if insufficient information is supplied to calculate $F(\omega)$ and $\theta(\omega)$ we know that $f(t)$ is expressible as an integral of this form. From formula (358) we have

$$\frac{1}{T}\int_0^T f^2\,dt=\frac{1}{\pi T}\int_0^\infty F^2\,d\omega.$$

Now let the epoch T become indefinitely large; if the disturbance is random in character, the left hand side of this equation will approach a limit, equal to the mean square value of $f(t)$, and we can write

$$f_m^2=\frac{1}{\pi}\int_0^\infty R(\omega)d\omega \qquad (360)$$

where $R(\omega)$ is the limit of $\frac{1}{T}F^2\,(\omega)$ as T becomes very large. $R(\omega)$ will be termed the *statistical frequency spectrum* of the disturbance.

Corresponding to the above the mean square current absorbed by the network is given by (See equ. (357))

$$I_m^2 = \frac{1}{\pi}\int_0^\infty \frac{R(\omega)}{|Z(i\omega)^2|}d\omega. \tag{361}$$

Formula (361) suggests a method for the determination of the frequency spectrum of the disturbance as follows: Let us suppose that the network is a selective circuit which absorbs current only in a narrow frequency band corresponding to $\omega_1 < \omega < \omega_2$. Then if $\omega_2 - \omega_1$ is made sufficiently small, formula (361) becomes

$$I_m^2 = \frac{1}{\pi} R(\omega_m) \int_0^\infty \frac{d\omega}{|Z(i\omega)|^2} \tag{362}$$

where ω_m is defined by the equation

$$\omega_m = \int_0^\infty \frac{\omega . d\omega}{|Z(i\omega)|^2} \div \int_0^\infty \frac{d\omega}{|Z(i\omega)|^2}. \tag{363}$$

It follows from the preceding that measurement of the mean square current and calculation or measurement of the impedance characteristics of the selective circuit supplies the information necessary to compute the statistical frequency spectrum of the disturbance.

Returning to formula (361) let us suppose that the selective circuit is designed to receive a band of frequencies $\omega_1 < \omega < \omega_2$ and excludes all other frequencies. The mean square current is then given by

$$I_m^2 = \frac{1}{\pi}\int_{\omega_1}^{\omega_2} \frac{R(\omega)}{|Z(i\omega)|^2}d\omega \tag{364}$$

$$= \frac{1}{\pi} R(\omega') \int_{\omega_1}^{\omega_2} \frac{d\omega}{|Z(i\omega)|^2} \tag{365}$$

where $\omega_1 < \omega' < \omega_2$.

If therefore the selective circuit is required to receive a band of frequencies corresponding to $\omega_1 < \omega < \omega_2$, formulas (364) and (365) give the irreducible minimum of interference. That is, the selective circuit must necessarily receive the energy of the disturbance which lies in the frequency range it is designed to receive. A definite limit is therefore set to the protection against interference which can be obtained by means of selective circuits.

This limitation is inherent in the nature of the interference and the frequency requirements of the signalling system.

For further and more detailed application of formulas (357) and (358) the reader is referred to the following papers:

Transient Oscillations in Electric Wave Filters, Bell System Technical Journal, July, 1923.

Selective Circuits and Static Interference, Trans. A. I. E. E., 1924.

An Application of the Periodogram to Wireless (Burch and Bloemsma), Phil. Mag., Feb., 1925.

The Theory of the Schrotteffekt (Fry), Journal Franklin Institute, Feb., 1925.

The foregoing must conclude our very brief account of the Fourier Integral and its applications in Electric Circuit theory; an adequate treatment of this subject would require a treatise in itself, and is beyond the scope of the present work. All that has been attempted is to give a very brief introduction to its significance in physical problems and a few of its outstanding applications in circuit theory. The reader who is interested in pursuing this subject further is referred to a paper by T. C. Fry on "The Solution of Circuit Problems" (Phys. Rev., Aug., 1919), which gives a rigorous discussion of the solution of the Fourier Integral by contour integration, together with some general forms of solution of the circuit problems.

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A word may be said regarding the following bibliography. In the body of the book, references have for the most part been confined to original papers dealing with special subjects the treatment of which in the text is either brief or inadequate. The majority of the papers listed in the bibliography are either original contributions to, or valuable expositions of, the operational calculus, or else important applications leading to original results. In addition, there are included important papers on electric circuit theory and on special topics such as the wave filter. Papers have been omitted which in the opinion of the author are, while excellent in themselves, essentially straightforward applications of the operational calculus. No claim to completeness can be made, however, for the bibliography, and possibly important papers have been omitted inadvertently. The author would appreciate having his attention called to such omissions.

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